THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Automorphic forms and string theory

Small automorphic representations and non-perturbative effects

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To my family

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Abstract

This compilation thesis stems from a project with the purpose of determining non-perturbative contributions to scattering amplitudes in string theory carrying important information about instantons, black hole quantum states and M-theory.

The scattering amplitudes are functions on the moduli space invariant under the discrete U-duality group and this invariance is one of the defining properties of an automorphic form. In particular, the leading terms of the low-energy expansion of four-graviton scattering amplitudes in toroidal compactifications of type IIB string theory are captured by automorphic forms attached to small automorphic representations and their Fourier coefficients describe both perturbative and non-perturbative contributions.

In this thesis, Fourier coefficients of automorphic forms attached to small automorphic representations of higher-rank groups are computed with respect to different unipotent subgroups allowing for the study of different types of non-perturbative effects. The analysis makes extensive use of the vanishing properties obtained from supersymmetry described by the global wave-front set of the automorphic representation.

Specifically, expressions for Fourier coefficients of automorphic forms attached to a minimal or next-to-minimal automorphic representation of SL_n, with respect to the unipotent radicals of maximal parabolic subgroups, are presented in terms of degenerate Whittaker coefficients. Additionally, it is shown how such an automorphic form is completely determined by these Whittaker coefficients.

The thesis also includes some partial results for automorphic forms attached to small automorphic representations of E_6, E_7 and E_8.

Keywords: string theory, automorphic forms and representations, U-duality, non-perturbative effects, instantons, Eisenstein series.

List of publications

This thesis is based on the following appended papers referred to by Roman numerals in the text.

References to sections, equations, figures or theorems of these papers will be denoted by the paper number followed by a dash and the reference number. For example, Paper I defines automorphic forms in section I-4.3.

Paper I

Eisenstein series and automorphic representations
Philipp Fleig, HG, Axel Kleinschmidt, Daniel Persson
Cambridge Studies in Advanced Mathematics, Cambridge University Press (2018)
ISBN 9781107189928, `http://dx.doi.org/10.1017/CBO9781316995860`
In preparation (final stage of editing). Appearing here by permission.
`arXiv:1511.04265[math.NT]`

Because of page restrictions, only Part 1 of 3 is included in the printed thesis.

Paper II

Small automorphic representations and degenerate Whittaker vectors
HG, Axel Kleinschmidt, Daniel Persson
Journal of Number Theory 166 (Sep, 2016) 344–399
`https://doi.org/10.1016/j.jnt.2016.02.002`
`arXiv:1412.5625[math.NT]`

Paper III

Fourier coefficients attached to small automorphic representations of $SL_n(\mathbb{A})$
Olof Ahlén, HG, Axel Kleinschmidt, Baiying Liu, Daniel Persson
`arXiv:1707.08937[math.RT]`

Also based on a current project which will be described, but not appended:

Whittaker-Fourier coefficients and next-to-minimal automorphic representations
Dmitry Gourevitch, HG, Axel Kleinschmidt, Daniel Persson, Siddhartha Sahi
Preliminary title. Work in progress.

Contribution report

For the list publications above, authors are listed alphabetically and have contributed equally to discussions, theory, computations, writing and proofreading.

Paper I

I was involved in the preparation, production and writing of all portions of the book, but with main focus on Part 1 and Part 2. Because of the collaborative work between authors spanning over all chapters, the extent of the text, and the many revisions, it is difficult to list sections attributed to a single author, but I was, in particular, a driving force behind the sections about Lie algebras, parabolic subgroups, p-adic and adelic characters, topology of the adeles, strong approximation, adelisation of Eisenstein series, Eisenstein series for minimal and non-minimal parabolic subgroups, the restriction of adelic Fourier coefficients, the SL_3 Whittaker coefficient examples, relations between Fourier coefficients and Whittaker coefficients, string theory concepts, non-perturbative corrections from instantons, supergravity- and D-instantons, supersymmetry constraints in ten dimensions, and classical theta series among others. I have also been essential in making the book more pedagogical, clear and rigorous.

Paper II

I was involved in the preparation, production and writing of all parts of the paper, I stated and proved Theorems I–IV and did the computations for SL_3 and SL_4.

Paper III

I was involved in the preparation, production and writing of all parts of the paper. I, Baiying and Olof discussed the proofs of the main theorems together, but I was responsible for the details of Theorem A and the rank one case of Theorem B together with stating and proving the related lemmas. I wrote the appendix about Levi orbits and was the driving force behind the theory section about nilpotent orbits and Fourier coefficients.

For the listed publications I have also had a prominent rôle in handling revision control of the project source files, software development and computations as well as drawing many of the illustrations.

Acknowledgements

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Thank you Daniel Bump, Minhyong Kim, Stephen D. Miller and Martin Westerholt-Raum for your interesting discussions regarding this thesis and the appended papers.

For your excellent company and friendship, thank you to all my current and former colleagues at the Division of Theoretical Physics and the former Department of Fundamental Physics.

Last, but not least, I am incredibly grateful to my wonderful friends and family for all their love and support.

Thank you!

Contents

Chapter 1

Introduction

The unification of gravity and quantum theory has long been one of the most challenging problems in theoretical physics, as standard methods for quantising Einstein's classical theory of gravity yield divergences and unphysical results.

Taking a completely different approach, string theory is a consistent theory of quantum gravity with strings, instead of point particles, as fundamental objects, and the theory incorporates both gravity and particle physics.

However, string theory is not completely understood beyond a perturbative description expanding around weak interactions. The non-perturbative behaviour can be used to understand instantons, M-theory and the quantum states of black holes.

The purpose of this thesis is to develop methods for computing Fourier coefficients of automorphic forms attached to small automorphic representations on higher-rank groups, which is part of a long-term goal to determine non-perturbative contributions to string scattering amplitudes using constraints from U-duality and supersymmetry.

In this chapter we will first review the need of a theory of quantum gravity, introduce a few concepts (treated more thoroughly in later chapters) needed to clarify the aim and objectives of the thesis, summarise the main results, which are described in more detail in chapter 4, and end with a reader's guide for the remaining chapters of the thesis.

1.1 Quantum gravity

Research in theoretical high energy physics aims to understand the fundamental principles and laws that govern our universe. Two very successful theories of the 20th century capturing such principles are the *standard model of particles physics*, that describes the interaction between elementary particles in the framework of quantum field theory, and *Einstein's theory of general relativity*, which is used to describe gravity and its interaction with matter. They have both been thoroughly tested by experiments to very high accuracies, made several predictions that were later verified, and together they cover all four of the known fundamental forces: gravitational, electro-magnetic, and the weak and strong nuclear forces.

There are, however, several reasons for why these two theories do not give a complete picture, two of the perhaps most evident being:

– The standard model of particle physics does not include dark matter.

– Einstein's theory of gravity breaks down close to singularities, such as black holes and the Big Bang.

In a complete physical description of the universe these items must be addressed. Indeed, there are plenty of observational evidence for the existence of dark matter from studying distant galaxies [1]. Black holes have recently been indirectly detected by the LIGO experiment measuring the gravitational waves transmitted from two rapidly rotating black holes [2].

The two theories are also complementary in the meaning that the standard model does not include gravitational interactions while Einstein's theory of gravity lacks interactions from the remaining fundamental forces. The gravitational force is usually much weaker than the other fundamental forces, which is why this separation has been possible and enjoyed such success in the past. Similarly, because of its weak strength, quantum effects in gravity are only expected to emerge at very high energies or short length scales close to the Planck length.

But there are situations where they cannot successfully be treated separately. The close proximity of the singularity of a black hole is such an example, where the effects from both gravity and quantum mechanics are strong, which makes black holes important objects to study in quantum gravity.

Although string theory can also be used to model particle physics beyond the standard model, this is will not be addressed in this thesis.

It is one of the currently most challenging problems in theoretical physics to unify quantum mechanics and gravity – to obtain a *quantum theory of gravity*. Such a theory is expected to resolve the black hole singularity and the classical theory would then be seen as an effective theory for length scales larger than the Planck length. Although not a requirement of quantum gravity, it is hoped to also unify all the fundamental forces in a single framework.

Early attempts to quantise the gravitational field, in a way similar to that of the fields for the remaining forces, failed, leading to unphysical results and divergences [3]. A theory which does not yield finite results at large energies (or, equivalently, small length scales) is said to be *UV-divergent*, where UV refers to ultraviolet or high energies, and such a divergence indication that, at a larger energy scale, new physical phenomena enters requiring a more detailed theory – a theory that reduces to what is already known for lower energies.

Since the canonical method of quantisation is not applicable to gravity, a completely different approach is needed. String theory is such an approach where one, instead of point particles, considers strings as fundamental objects moving in space-time. The vibrational fluctuations of the string can be quantised and the different modes give rise to different particle types, a spectrum which includes the graviton. String theory is therefore a quantum theory of gravity and gives high-energy corrections to Einstein's theory of general relativity.

1.2 Background and previous research

String theory is a very large area of research with many subjects beyond the scope of this thesis. In chapter 2, some fundamental concepts of string theory are introduced followed by a more detailed motivation for studying automorphic forms in string theory discussing non-perturbative contributions to four-graviton scattering amplitudes based on [4–26]

In this section, the most relevant background and previous research in connection with this thesis is very briefly introduced to better be able to describe the aims and objectives of the thesis in section 1.3.

The moduli space of type IIB string theory in ten dimensions is described by the coset space $SL_2(\mathbb{R})/SO_2(\mathbb{R})$ [27,28] which is homeomorphic to the Poincaré upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. The theory enjoys a discrete $SL_2(\mathbb{Z})$-symmetry called U-duality, which means that physical observables, such as scattering amplitudes, are functions on the moduli space invariant under this discrete subgroup [4, 29]. If the physical observables further satisfy some differential equations and growth conditions detailed in section 3.3, they fall into a class of functions called automorphic forms, with perhaps the most well-known examples of automorphic forms being non-holomorphic Eisenstein series.

In [4,6,9,10] it was discovered that the leading-order low-energy (or α') correction to the four-graviton scattering amplitude in ten-dimensional type IIB string theory, the so called R^4-term[1], is captured by a non-holomorphic Eisenstein series on $\mathbb{H}$.

In fact, supersymmetry constraints imply [8] that the R^4 coefficient is an eigenfunction to the Laplacian on the Poincaré upper half plane with an eigenvalue matching the above Eisenstein series suggested in [4] and, it was shown in [9], that the solution to this eigenfunction equation, constrained by the known asymptotic behavior obtained from string perturbation theory, is unique.

The coefficient to the next higher-order term, D^4R^4, was later found to be another Eisenstein series in [30]. However, the third correction, D^6R^4, which was obtained in [31], is not an Eisenstein series, nor an automorphic form in a strict sense since it does not satisfy the Z-finiteness condition in definition 3.1.

Apart from the four-graviton interactions, similar progress has also been made for fermionic terms as seen in [32] for the λ^{16}-term.

When compactifying on tori to lower dimensions, the moduli space becomes larger and can be expressed as a coset space $G(\mathbb{R})/K(\mathbb{R})$ of a group $G(\mathbb{R})$ and its maximal compact subgroup $K(\mathbb{R})$ [33, 34], where the different groups and the corresponding U-duality groups $G(\mathbb{Z})$ [29,35] are shown in table 2.2. The scattering amplitudes and their coefficient functions on these toroidal compactifications were studied in [6, 9–13, 18, 20, 22, 36] with supersymmetry constraints obtained in [37–41].

It was shown in [14, 15, 17, 18] that the automorphic forms for the first two α'-corrections, that is, the R^4 and D^4R^4 coefficients, are attached to a minimal and a next-to-minimal automorphic representation respectively. This means that their Fourier coefficients are heavily constrained by the *global wave-front set* [18, 42].

The above D^nR^4 coefficients contain all perturbative and non-perturbative

[1] The notation R^4 will be explained in section 2.4

contributions in the string coupling g_s at each of the corresponding orders in α' and are expressed as infinite sums over cosets. To extract physical information from these functions, one needs to compute their Fourier coefficients with respect to the unipotent radical of different parabolic subgroups of G [12], corresponding to studying different limits of the string theory shown in table 2.3. An important example is the decompactification limit where one can extract information about higher dimensional black holes. The zero-modes, or constant terms, of the Fourier expansion give the perturbative corrections (with respect to the corresponding limit) while the remaining modes give non-perturbative corrections.

For the R^4 and D^4R^4-terms in ten dimensions, the Fourier expansion of the relevant automorphic forms were known in the literature and could be readily computed using Poisson resummation, and the fact that the moduli space is homeomorphic to the Poincaré upper half plane $SL_2(\mathbb{R})/SO_2(\mathbb{R}) \cong \mathbb{H}$.

However, for lower dimensions, where the groups $G(\mathbb{R})$ have higher rank, these simplifying circumstances are no longer always available and the Fourier coefficients have only been computed in a few specific cases as in, for example, [43]. There, the Fourier coefficients for the R^4 and D^4R^4 coefficients were, in various dimensions, computed with respect to the unipotent subgroup associated to the decompactification limit discussed in section 2.4.4. For the Fourier coefficients in other limits for the R^4 and D^4R^4 interactions in dimensions $6 \leq D \leq 8$, see [18].

In this thesis we use adelic methods for computing Fourier coefficients on $G(\mathbb{R})$. As illustrated in figure 1.1, and discussed in detail in chapter 3, we lift the functions on the moduli space $G(\mathbb{R})/K(\mathbb{R})$ to functions on $G(\mathbb{A})$. We can then compute the adelic Fourier coefficients of the lifted functions and restrict the argument to recover the Fourier coefficients on $G(\mathbb{R})$ which are the objects of interest in string theory.

Our methods for determining the Fourier coefficients of automorphic forms attached to small automorphic representations are inspired by [44] which showed that an automorphic form attached to a minimal automorphic representations of E_6 or E_7 is completely determined by maximally degenerate Whittaker coefficients as well as the constant mode.

Paper II is also inspired by [45] for the construction of Fourier coefficients associated to nilpotent orbits. The method for computing the Fourier expansion of an automorphic form attached to a small automorphic representation of SL_n in Paper III uses similar row and column expansion methods as Piatetski-Shapiro and Shalika in [46, 47] which determined the Fourier expansions of cusp forms on

$$
\begin{tikzcd}
\text{Function on } G(\mathbb{A}) \arrow[r, "\text{compute}"] & \text{Fourier coefficient on } G(\mathbb{A}) \arrow[d, "\text{restrict}"] \\
\text{Function on } G(\mathbb{R}) \arrow[u, "\text{lift}"] & \text{Fourier coefficient on } G(\mathbb{R})
\end{tikzcd}
$$

Figure 1.1: Illustration of the computation of Fourier coefficients on $G(\mathbb{R})$ by first lifting to the adeles.

GL_n. For expansions of non-cuspidal automorphic forms on GL_n, see [48, 49]. In Paper III, we also compare maximal parabolic Fourier coefficients of automorphic forms attached to small automorphic representations of SL_5 of interest in string theory with previous results from [18] obtained by different methods.

In mathematics, automorphic forms relate number theory, representation theory and algebraic geometry with a web of conjectures that are collectively called the Langlands program [50, 51]. In particular, Fourier coefficients of automorphic forms are a source of L-functions which are, roughly speaking, analytic functions used to study different arithmetic properties.

1.3 Aim and objectives

While the overarching goal in the community is to obtain a detailed non-perturbative description of string theory, this thesis is part of a long-term project with the aim to determine and understand the constraints on (non-perturbative contributions to) string scattering amplitudes from U-duality and continuous symmetries.

> Specifically, this amounts to
>
> *Computing Fourier coefficients of automorphic forms attached to small automorphic representations with respect to the unipotent radicals of maximal parabolic subgroups*
>
> which is the main objective of this thesis.

This is put in context by the objectives for the long-term project which are to use the theory of automorphic forms to determine the first order α'-corrections to scattering amplitudes, such as the four-graviton amplitudes in toroidal compactifications of type IIB string theory, and to compute their Fourier coefficients with respect to the unipotent radicals of certain maximal parabolic subgroups to obtain the perturbative and non-perturbative corrections in the corresponding expansion variables. This would enable the study of higher dimensional BPS-states, D-instantons, and M2- and M5-instantons.

As seen in section 1.2, there has been a lot of progress towards these objectives by the work of many authors (referenced there). For example, the R^4 and D^4R^4 contributions have been determined for dimensions $3 \leq D \leq 10$ as automorphic forms, and their Fourier coefficients have been computed in certain cases. Additionally, the D^6R^4 correction, which is not strictly and automorphic form, and its Fourier coefficients are known in ten dimensions.

As a natural continuation of these developments, it remains to compute the Fourier coefficients of the known automorphic forms with respect to all the maximal parabolic subgroups of interest, and to determine the D^6R^4 corrections for lower dimensions. The subsequent α'-corrections are not expected to be equally constrained by supersymmetry, which makes it more difficult to determine them and to compute their Fourier coefficients.

1.4 Summary of results

Below follows a brief summary of the main results presented in this thesis. For precise statements and further details, please see chapter 4.

Paper I gives a review of the necessary theory and background for studying automorphic forms including the computation of *Whittaker coefficients* (which are Fourier coefficients with respect to the unipotent radical of a Borel subgroup) of spherical Eisenstein series using *the Langlands constant term formula* and *the Casselman-Shalika formula*. It covers many examples and applications and will be an important supporting text to chapter 3.

In section 4.1 methods for computing first trivial, unramified, generic and then degenerate Whittaker coefficients are presented, gathered and extended from the existing literature. These are then the starting point for computing more general Fourier coefficients with respect to (the unipotent radical of) *maximal parabolic subgroups* which are of importance in string theory as discussed in section 2.4.4.

As mentioned in section 1.2, the automorphic forms of interest in string theory are attached to small automorphic representations which have, in several cases, been proven to be completely determined by degenerate Whittaker coefficients [44]. Note that an expansion for SL_2 is trivially expressed in terms of maximally degenerate Whittaker coefficients.

Paper II shows, for SL_3 and SL_4, how an automorphic form and its Fourier coefficient can be expressed in terms of Whittaker coefficients. In particular, automorphic forms attached to a minimal or next-to-minimal automorphic representation of these groups are expressed in terms of Whittaker coefficients supported on at most a single simple root or Whittaker coefficients supported on at most two commuting[2] roots respectively using a construction from [45] as summarised in section 4.3. Similar results for maximal parabolic Fourier coefficients were also obtained in Paper II and are shown in section 4.2.

Paper III generalises these ideas for SL_n, $n \geq 5$, but uses the notion of Whittaker pairs and methods based on [42], as well as a lemma for exchanging roots in Fourier integrals from [52]. Similar to the case for SL_3 and SL_4, the automorphic forms for SL_n and their maximal parabolic Fourier coefficients are also expressed in terms of Whittaker coefficients supported on at most a single root or at most two commuting roots for a minimal and next-to-minimal automorphic representation respectively.

Encouraged by these findings, Paper II also investigates whether the maximal parabolic Fourier coefficients of automorphic forms in a minimal automorphic representation of E_6, E_7 or E_8 are similarly expressed in terms of maximally degenerate Whittaker coefficients by comparing the local factors of such Whittaker coefficients with known local spherical vectors of[3] $\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \psi_p$ computed in [14, 53–55] for several different maximal parabolic subgroups P using methods from representation theory. The complete agreement provides strong support for that the property for maximal parabolic Fourier coefficient proven for SL_n in Papers II and III also holds for these groups.

[2]In the meaning that their Chevalley generators E_α commute.

[3]The notation is explained in section 3.5.

1.5 Reader's guide

The thesis is organised as follows. Chapter 2 gives an introduction to string theory concepts and, in particular, the coefficients in the derivative expansion of four-graviton scattering amplitudes in toroidal compactifications of type IIB string theory. The coefficients are functions on the string moduli space and from supersymmetry and U-duality we obtain constraints for these functions. We study instanton contributions and show that the leading-order coefficients in the derivative expansion is determined by Eisenstein series. The chapter is concluded by a discussion of the physical interpretation of their Fourier coefficients in different limits of string theory.

Motivated by this, chapter 3 introduces the theory of automorphic forms and representations on adelic groups, and, in particular Eisenstein series. We discuss their Fourier expansion and the vanishing properties of the Fourier coefficients using the global wave-front set.

Chapter 4 presents the main theorems of Papers I–III together with a brief discussion of the structure of their proofs. We also relate the results between the different papers and to the overarching theme of the thesis.

Chapter 5 concludes with a discussion and outlook connecting the results with the aims and objectives of the introduction in chapter 1. In particular, we discuss an ongoing project for developing methods to compute maximal parabolic Fourier coefficients of automorphic forms attached to small automorphic representations of the exceptional groups.

Chapter 2

String theory concepts

This chapter motivates the study of automorphic forms and their Fourier coefficients by highlighting their important rôle in string theory scattering amplitudes. We will, however, in this thesis only be able to scratch the surface of this large area of research. For further details please see Part 2 of Paper I.

2.1 Supersymmetry and supergravity

Following [56], we will now discuss supersymmetry and supergravity in ten dimensions since we will use the ten-dimensional case as an example throughout the thesis.

The supersymmetry, or super-Poincaré, algebra is similar in other dimensions, but the Clifford algebra representations may vary. The Clifford algebra associated with the Lorentz group in D dimensions is described by generators γ^μ where $\mu = 0, \ldots, D-1$ satisfying

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}\mathbb{1} \tag{2.1}$$

where and $\eta^{\mu\nu}$ is the Minkowski metric $\operatorname{diag}(-1, 1, \ldots, 1)$. The commutators $\Sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ are generators of a representation of the Lorentz algebra $SO(D-1, 1)$ [56].

In curved space-time, local Lorentz frames and vielbeins are used to introduce a Clifford algebra, but we will mainly work with small perturbations around a Minkowski background and therefore refer the reader to [56, Chapter 7] for a general treatment.

For dimensions $D = 2m$ and $D = 2m+1$, a 2^m-dimensional *Dirac representation* of the Clifford algebra is constructed in [56, Section 3.1] as complex matrices and the elements in the module are called *Dirac spinors* (which we saw also furnish a representation of the Lorentz algebra).

The complete Clifford algebra consists of the identity $\mathbb{1}$ and all (anti-symmetric) products formed by the generators. Because of the anti-symmetric properties, there exists a unique highest rank Clifford algebra element (up to normalisation)

$$\gamma_* = (-i)^{m+1} \gamma_0 \gamma_1 \cdots \gamma_{D-1} \,. \tag{2.2}$$

Since $\gamma_*^2 = \mathbb{1}$ we may define the projection operators

$$P_L = \frac{1}{2}(\mathbb{1} + \gamma_*) \qquad P_R = \frac{1}{2}(\mathbb{1} - \gamma_*) \tag{2.3}$$

Table 2.1: Super-Poincaré generators in ten dimensions with degree (or statistics) and Lorentz representation.

	Generator	Degree/statistics	Lorentz representation
P_μ	translations	bosonic	vector
$M_{\mu\nu}$	rotations and boosts	bosonic	traceless symmetric matrix
Q_α	supersymmetry	fermionic	Majorana-Weyl spinor

where the eigenvalue for γ_* is called *chirality* and the projected spinors are called *chiral spinors* being either left-handed or right-handed as denoted by the subscripts of the projection operators.

It is shown in [56] that, for even dimensions with $D = 2m$, the Dirac representation is reducible and the two subrepresentations, called *Weyl representations*, are given by the chirality projection operators above. Secondly, for $D = 2, 3, 4 \pmod 8$ the Dirac representation can be made real, reducing the number of independent components, and the corresponding spinors are called *Majorana spinors*. Lastly, for $D = 2 \pmod 8$ the Majorana reality condition is compatible with the chiral projection allowing for *Majorana-Weyl spinors*.

In ten dimensions, the Majorana-Weyl spinors are, in some sense, the most elementary spinors available with $2^{m-1} = 16$ independent, real components and are used to describe ten-dimensional supergravity.

Even if we choose to not work in a real representation for $D = 2, 3, 4 \pmod 8$, we can still talk about a Majorana condition, which then, instead of being a reality condition of a spinor ψ, is described by an anti-symmetric charge conjugation matrix C (determined by the transpose properties of the Clifford generators) as [56]

$$\psi^* = iC\gamma^0\psi \,. \tag{2.4}$$

The charge conjugation matrix can in this way be used relate a Majorana spinor (usually denoted with a lower spinor index ψ_α) with its dual in the conjugate representation (usually denoted with an upper spinor index ψ^α). In a similar way, the indices of γ-matrices, as in for example $(\gamma^\mu)_\alpha{}^\beta$ can be raised or lowered with the charge conjugation matrix.

The super-Poincaré algebra is a $\mathbb{Z}_2$-graded algebra where odd elements are called *fermionic* and even elements *bosonic*. A commutator is usually denoted by $[\cdot, \cdot]$ and an anti-commutator by $\{\cdot, \cdot\}$.

That we consider a $\mathbb{Z}_2$-graded algebra is motivated by the restrictions on how space-time symmetries can be combined with internal symmetries from the Coleman–Mandula theorem (and its generalisation: the Haag–Lopuszanski–Sohnius theorem) [57].

The generators of the super-Poincaré algebra in ten dimensions are listed in table 2.1 where, in particular, we have that the generator of supersymmetry transformations, called Q_α, is a Majorana-Weyl spinor. It is possible to have several copies of the supersymmetry generator Q_α and the number of supersymmetry generators is denoted by $\mathcal{N}$. In the case of chiral generators, $\mathcal{N}$ is usually written as a tuple of the number of left chiral and right chiral generators.

The maximal number of supercharges, that is, the total number of real components of the supersymmetry generators, is 32 for an interacting theory in flat space-time without an infinite tower of fields with spin larger than two [58]. This means that the largest dimension for which there exists a supergravity theory is $D = 11$ since above that the spinors have more than 32 components.

In ten dimensions, the Majorana-Weyl spinors Q_α have 16 independent real components. Thus, there are three[4] possible supergravity theories in ten-dimensions with the following names and supersymmetry generators:

1. **Type I:** one chiral supersymmetry generator.
2. **Type IIA:** two chiral supersymmetry generators of opposite chirality.
3. **Type IIB:** two chiral supersymmetry generators of equal chirality.

The names match those of different ten-dimensional string theories. In fact, the low-energy effective actions (as discussed in section 2.2) of these three string theories are the respective supergravity theories listed above [59, 60].

The generators satisfy the usual Poincaré commutation relations as well as [56]

$$\begin{aligned} \{Q^i_\alpha, Q^j_\beta\} &= -\tfrac{1}{2}\delta^{ij}(\gamma_\mu)_{\alpha\beta}P^\mu \\ [M_{\mu\nu}, Q_\alpha] &= -\tfrac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta \\ [P_\mu, Q_\alpha] &= 0 \end{aligned} \tag{2.5}$$

where we have suppressed central charges which will be discussed below, lowered all spinor indices with the charge conjugation matrix, $i = 1, \ldots, \mathcal{N}$ and the γ-matrices with multiple indices are anti-symmetrised products of the Clifford algebra generators γ^μ.

Particles are characterised by unitary irreducible Poincaré representations (induced from the *Wigner little group* which is the stabiliser subgroup of a representative momentum vector) according to the *Wigner classification*. Representations of the super-Poincaré algebra are *supermultiplets* of particles that transform into each other under supersymmetry transformations by Q_α. Since $[P_\mu, Q_\alpha] = 0$, the operator $m = -P_\mu P^\mu$ is a Casimir operator of the super-Poincaré algebra, interpreted as rest mass, which means that all particles in the supermultiplet have the same mass.

The representations of the superalgebra are also $\mathbb{Z}_2$-graded spaces such that the odd operator Q_α changes the degree (fermionic or bosonic) of the field or state they act on. For any representation in which P_μ is a one-to-one operator, the bosonic and fermionic subspaces are of the same dimension since, schematically, $\{Q, Q\} = P$ [61]. This will be the case for the theories we consider in this thesis.

Supersymmetry transformations are parametrised by a spinor parameter ϵ_α and the infinitesimal change of a field $\Psi(x)$ under a supersymmetry transformation is

$$\delta\Psi(x) = \epsilon^\alpha Q_\alpha \cdot \Psi(x)\,. \tag{2.6}$$

[4] We will not include the heterotic supergravity theories here which are also coupled to super-Yang-Mills theory [59].

If ϵ_α is constant the supersymmetry transformation is called global, and if it is space-time dependent the transformation is called local.

Supergravity is a theory invariant under local supertransformations and since the anticommutator of supersymmetry generators is a translation, this means that the theory is invariant under local translations as well. Thus, supergravity is a diffeomorphism invariant theory and includes gravity. It is also true that a supersymmetric theory of gravity must have local supersymmetry [56].

The supersymmetry algebra may also have central charges Z that commute with the supercharges Q_α (but not necessarily with the Lorentz generators) and depend on electro-magnetic charges of the states they act on as well as certain scalar moduli parameters of the theory [62].

When the mass and central charges satisfy certain relations (a saturation of the BPS bound), a fraction of the 32 supercharges[5] may act trivially in the supersymmetry representation, making it a 'short' representation. The states of such a representation is called a *BPS-state* and is labeled by the fraction of supercharges that act trivially (for example $\frac{1}{2}$-BPS, $\frac{1}{4}$-BPS and $\frac{1}{8}$-BPS). These states are important for studying non-perturbative effects and dualities since the BPS bound is protected[6] from quantum corrections when varying interaction strengths, and a lot of information can be extrapolated from the weak coupling behaviour [59].

In string theory, D-branes are extended objects of different dimensions on which the endpoints of open strings can attach. D-branes are $\frac{1}{2}$-BPS states and are related to supergravity solutions which are invariant under half of the supersymmetry generators [64, 65].

We will in this thesis focus on the type IIB supergravity in ten dimensions and, as seen in the above list, the theory has two supersymmetry generators which are Majorana-Weyl spinors of the same chirality. The particle content of the theory is: two left-handed Majorana-Weyl gravitinos and two right-handed Majorana-Weyl dilatinos, which are superpartners to the graviton (described as fluctuations of the space-time metric G) and the dilaton ϕ which is a scalar. We also have a two-form B_2 with field strength $H_3 = dB_2$ and three Ramond-Ramond r-forms C_0, C_2 and C_4 with field strengths $F_{r+1} = dC_r$ [3, 66]. The scalar C_0 is the axion, which we will often denote as χ, not to be confused with the Euler characteristic $\chi(\Sigma)$ of a surface Σ discussed in section 2.2. All these particles correspond to the massless states of the type IIB superstring theory and are in the same supermultiplet [3, 59]. We also define the combinations

$$\begin{aligned}\tilde{F}_3 &= F_3 - C_0 H_3 \\ \tilde{F}_5 &= F_5 - \frac{1}{2} H_3 \wedge C_2 + \frac{1}{2} F_3 \wedge B_2\,.\end{aligned} \tag{2.7}$$

To obtain the correct degrees of freedom, the combined field strength $\tilde{F}_5$ is required to be self-dual $*\tilde{F}_5 = \tilde{F}_5$. However, the standard type IIB supergravity action presented below will not enforce this from its equations of motions, which

[5] or the associated creation and annihilation operators

[6] There may however be discontinuous changes across certain walls in the moduli space due to a phenomenon called wall-crossing [63].

means that one has to manually impose the self-duality constraints on the solutions afterwards [3, 28].

The difficulties of constructing an action which does not require the self-duality constraint to be imposed manually is discussed in [3, 28]. In [67], such an action is formulated, but with extra free fields that decouple from the remaining part of the theory. Earlier attempts, reviewed in [67], have similar drawbacks such as the loss of manifest Lorentz invariance or an infinite number of auxiliary fields.

The bosonic part of the action is split up into three parts: the Neveu-Schwarz, Ramond and Chern-Simons terms, here shown in the *string frame* [28, 66]

$$
\begin{aligned}
S_{\text{SUGRA}} &= S_{NS} + S_R + S_{CS} \\
S_{NS} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G^{(S)}}e^{-2\phi}\Big(R^{(S)} + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2\Big) \\
S_R &= -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G^{(S)}}\Big(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\Big) \\
S_{CS} &= -\frac{1}{4\kappa_{10}^2}\int C_4 \wedge H_3 \wedge F_3
\end{aligned}
\tag{2.8}
$$

where $G^{(S)}$ and $R^{(S)}$ are the space-time metric and curvature scalar respectively in the string frame, and κ_{10} is the gravitational coupling constant.

By switching to the *Einstein frame*, with

$$
G_{\mu\nu} = G^{(E)}_{\mu\nu} = e^{-\phi/2}G^{(S)}_{\mu\nu} \tag{2.9}
$$

and correspondingly transform the curvature scalar R, we will see that this action has an $SL_2(\mathbb{R})$ symmetry as follows. Let $z = C_0 + ie^{-\phi} = \chi + ie^{-i\phi}$ be the axio-dilaton and

$$
G_3 = \tilde{F}_3 - ie^{-\phi}H_3 = F_3 - \tau H_3\,. \tag{2.10}
$$

Then the bosonic part of the type IIB supergravity action in the Einstein frame becomes [59]

$$
\begin{aligned}
S_{\text{SUGRA}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\Big(R - \frac{\partial_\mu z\partial^\mu z}{2(\operatorname{Im} z)^2} - \frac{1}{2}\frac{|G_3|^2}{\operatorname{Im} z} - \frac{1}{4}|\tilde{F}_5|^2\Big) + \\
+ \frac{1}{8i\kappa_{10}^2}\int \frac{1}{\operatorname{Im} z}C_4 \wedge G_3 \wedge \tilde{G}_3\,,
\end{aligned}
\tag{2.11}
$$

which we see is invariant under the following $SL_2(\mathbb{R})$ transformations

$$
z \to \gamma(z) = \frac{az+b}{cz+d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \tag{2.12}
$$

$$
\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \qquad G_{\mu\nu} \to G_{\mu\nu} \qquad \tilde{F}_5 \to \tilde{F}_5\,. \tag{2.13}
$$

As we will see in section 2.3, this symmetry will break to the discrete subgroup $SL_2(\mathbb{Z})$ when we also take quantum effects into account.

Before moving on to the next section about string theory let us review another way of describing supersymmetric field theories using *superspace*.

The physical fields of the ten-dimensional type IIB supergravity can be packaged into a generating function called a superfield which is a function of the space-time coordinates as well as complex Grassmann variables θ^α, $\alpha = 1, \ldots, 16$ together transforming as a Weyl spinor under $Spin_{9,1}$. Together, the coordinates parametrise a superspace and the physical fields (or component fields) are coefficient in the θ-expansion of the superfield which consists of finitely many terms [68].

The supersymmetry transformations generated by Q_α are then translations on superspace. Specifically, [69]

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} \qquad Q^*_\alpha = -\frac{\partial}{\partial \theta^\alpha} + 2i(\overline{\theta}\gamma^\mu)_\alpha \frac{\partial}{\partial x^\mu} \tag{2.14}$$

where $\overline{\theta} = \theta^\dagger \gamma^0$ is the Dirac conjugate. The operators Q_α and Q^*_α anticommute with the following covariant derivatives

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + 2i(\gamma^\mu \theta^*)_\alpha \qquad D^*_\alpha = -\frac{\partial}{\partial \theta^{*\alpha}} \,. \tag{2.15}$$

Note that we here use a single complex Weyl spinor Q_α with 16 complex components instead of two Majorana-Weyl spinors with 16 real components each. Also, the generator Q_α changes the powers of the θ^α coordinates in the expansion of the superfield and thus relates bosons, whose component fields have an even number of free spinor indices, and fermions, which have an odd number of free spinor indices.

In section 2.4.1, we will discuss the constraints for the superfield describing the degrees of freedom of the ten-dimensional type IIB supergravity as well as construct a (linearised) supergravity action by integrating a function F of the superfield.

2.2 String theory

The ten-dimensional supergravity theories discussed above are low-energy effective actions of the corresponding superstring theories. These string theories are finite for large energies and give higher-order corrections to the supergravity theories [3]. Let us now introduce some string theory concepts that will be important for the remaining part of the thesis. We will, for simplicity, mainly focus on the bosonic string theory, but many principles and features carry over to the supersymmetric case. Elsewhere in this thesis we will almost exclusively consider superstring theory which we will simply call string theory. For a more complete review of string theory see [3, 27, 28, 59]. Further details can also be found in chapter I-12.

Instead of point particles, the fundamental objects in string theory are strings that can be closed or open. Much like a particle sweeps out a trajectory over time, strings sweep out a surface called a world-sheet in space-time. Let Σ and M be manifolds of dimensions two and D which will describe the world-sheet and space-time respectively. String theory is essentially a theory describing the dynamics for the embedding maps $X : \Sigma \to M$ where M is also called the target-space. It is often convenient to work with the coordinate maps X^μ of X where $\mu = 0, \ldots, D-1$ and we let σ^m with $m = 0, 1$ be coordinates on Σ. One often starts with manifolds and metrics with Euclidean signature, making an analytical continuation in the time coordinate to obtain physical quantities in a Lorentzian signature.

Endowing M with a metric $G = G_{\mu\nu} dx^\mu \otimes dx^\nu$, this induces a metric γ on Σ

$$\gamma = \gamma_{mn} d\sigma^m \otimes d\sigma^n = G_{\mu\nu}(X) \frac{\partial X^\mu}{\partial \sigma^m} \frac{\partial X^\nu}{\partial \sigma^n} d\sigma^m \otimes d\sigma^n \tag{2.16}$$

and the string theory dynamics can be described by the *Nambu-Goto action* which measures the area of Σ embedded in M

$$\text{Area}(\Sigma) = \int_\Sigma d^2\sigma \sqrt{\det \gamma}\,. \tag{2.17}$$

However, because of the square root, this action is not easily quantised, which is why we introduce an auxiliary metric $g = g_{mn} d\sigma^m \otimes d\sigma^n$ on Σ and use the *Polyakov action*

$$S[g, G, X] = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{g} g^{mn} \frac{\partial X^\mu}{\partial \sigma^m} \frac{\partial X^\nu}{\partial \sigma^n} G_{\mu\nu}(X) \tag{2.18}$$

which, after using the equations of motion for g_{mn}, is proportional to the Nambu-Goto action and is much easier to quantise. The parameter α' describes a typical area scale of the world-sheet embedded in space-time, and is related to the inverse string tension.

In addition to this action, we also have terms that govern the dynamics of other background fields such as the dilaton ϕ

$$S_{\text{dilaton}} = \frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{g} \phi(X) R^{(2)} \tag{2.19}$$

where $R^{(2)}$ is the curvature scalar on Σ obtained from the metric g_{mn}. If we separate the constant mode ϕ_0, which is usually chosen to be the asymptotic value at infinity, from the remaining dilaton field, this part of the dilaton action becomes topological and proportional to the Euler characteristic $\chi(\Sigma)$ of the world-sheet

$$\frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{g} \phi_0 R^{(2)} = \phi_0 \chi(\Sigma)\,. \tag{2.20}$$

Upon quantising the fluctuations of the classical string solutions to the equations of motion for (2.18), the creation operators for the different modes generate a spectrum of particles. The mass-squares for these particles are given by the mode numbers and separated by $1/\alpha'$ where the massless particles, for the closed string spectrum, include the graviton.

Strings interact by joining and splitting as shown in figure 2.1, the strength of which is governed by the string coupling constant $g_\text{s} = e^{\phi_0}$. Scattering amplitudes in string theory are computed by summing over all possible world-sheets weighted by e^{-S} from which we may separate the factor $e^{-\phi_0 \chi(\Sigma)} = g_\text{s}^{-\chi(\Sigma)}$ coming from (2.20). For closed strings (ignoring the punctures for external states for now since the

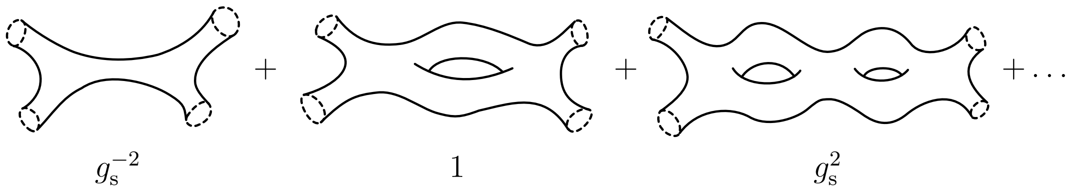


Figure 2.1: String world-sheets of genera $h = 0, 1, 2$ for scattering amplitudes of four gravitons together with their topological weights from the Euler characteristic. The dashed lines represent infinitely long legs for the external states.

corresponding g_s-factors are usually included in the vertex operators) where we do not have boundaries, a world-sheet of genus h is then weighted by $g_s^{2(h-1)}$. Thus, the sum over world-sheets can be organised into a sum over topologies each weighted by the string coupling constant to the negative power of the Euler characteristic, which becomes a power series in g_s and this procedure is called string perturbation theory.

The Polyakov action (2.18) is invariant under diffeomorphisms on Σ and M, as well as under Weyl transformations

$$g_{mn} \to e^{2\omega(\sigma)} g_{mn} , \tag{2.21}$$

where ω is a function on Σ. To obtain a consistent quantum theory without negatively normed states, the Weyl symmetry must not obtain an anomaly, that is, it should still be a symmetry of the quantum theory, and we will shortly see what this will imply in the path integral formalism.

In the scattering amplitude the (incoming and outgoing) external states are legs of the world-sheet stretched to infinity, as shown in figure 2.1, that can shrunk to punctures using the Weyl symmetry. The amplitudes can be computed using path integrals which are the sums over all possible world-sheet configurations weighted by the Polyakov action as e^{-S} discussed above. The punctures for external states appear in the path integral as certain vertex operators V_i which are different for different particle types and contain the external momenta and polarisations.

For closed bosonic strings the path integral for a scattering amplitude $\mathcal{A}$ with vertex operators $V_1, \ldots V_n$ takes the form [70, 71]

$$\mathcal{A} = \sum_{\text{genus } h} g_s^{2(h-1)} \int_{\text{Maps}(\Sigma,M)\times\text{Met}(\Sigma)} \frac{DXDg}{\text{Vol}(\text{Diff} \ltimes \text{Weyl})} V_1 \cdots V_n e^{-S[g,G,X]} \tag{2.22}$$

where $\text{Maps}(\Sigma, M)$ is the space of embeddings from the world-sheet Σ to the space-time M, $\text{Met}(\Sigma)$ is the space of all metrics g on Σ, and $S[g, G, X]$ is the Polyakov action. Because of the symmetries of the action we have divided by the volume of the group of diffeomorphisms and Weyl transformations to not overcount physically equivalent configurations.

For the Weyl symmetry to be a symmetry of the quantum theory, it is not enough for the Polyakov action S in (2.22) to be invariant under Weyl transformations; we also need the invariance of the path integral measure $DXDg$. The transformation of the measure depends on the dimension D of space-time, and the measure can be shown to be invariant only for a critical dimension $D = 26$ for bosonic string theory and $D = 10$ for superstring theory. In curved space-times M, the invariance of the measure will also put conditions on the space-time metric G and other background fields which in the former case are Einstein's equations for gravity with higher-order corrections from string theory [71].

We have mentioned before that the low-energy effective theory of the type IIB string theory is the type IIB supergravity theory. A low-energy effective action, which describes an ordinary quantum field theory, can be obtained by computing string theory scattering amplitudes, or one can use the equations of motion from the Weyl invariance of the path integral measure, and reverse engineer a field theory action which would obtain the same results [72].

An effective theory is only meant to be a good description for a certain energy or momentum range. Here we consider momenta which are small compared to $1/\sqrt{\alpha'}$ and we will see in section 2.4 that this amounts to making an expansion in α' where the leading-order contribution gives the supergravity theory. In section 2.4 we will also consider corrections to the supergravity effective action order by order in α' based on four-graviton scattering amplitudes

2.3 Toroidal compactifications and U-duality

If we want to study physics in dimensions lower than the critical dimension $D = 10$, we can consider different target-spaces M by letting X be a compact d dimensional manifold and $M = \mathbb{R}^{D-d} \times X$. To let the remaining theory in $\mathbb{R}^{D-d}$ still have a lot of supersymmetry one can for example let X be a d-torus T^d, contain a compact $K3$-surface $X = K3 \times T^2$, or be certain Calabi–Yau manifolds which preserve all, half or a quarter of the supersymmetry charges respectively. We will, in this thesis, mostly focus on toroidal compactifications since the large amount of preserved supersymmetry gives strong constraints on the scattering amplitudes.

Scattering amplitudes in string theory depend on data of the external states such as momenta and polarisations, but they also depend on the constant modes (or expectation values) of scalar fields in the theory which form the string moduli space.

In ten dimensions, the moduli space of the type IIB string theory is parametrised by the constant mode of the axio-dilaton $z = \chi_0 + i g_s^{-1}$ on the Poincaré upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$ which is homeomorphic to the coset space $SL_2(\mathbb{R})/SO_2(\mathbb{R})$. When compactifying, the moduli space becomes larger because of the added scalars and for toroidal compactifications the resulting moduli space is given by a coset space $G(\mathbb{R})/K(\mathbb{R})$ where the groups $G(\mathbb{R})$, and their maximal compact subgroups $K(\mathbb{R})$ are shown in table 2.2. All groups $G(\mathbb{R})$ are here considered to be associated with split real forms. The group G can also be visualised by adding nodes to the Dynkin diagram in figure 2.2 in the order of the Bourbaki labeling shown there. When comparing groups for different dimensions we will use the superscript notation $G^{(D)}$ for the group $G(\mathbb{R})$ of table 2.2 in D dimensions after compactifying on a d-torus T^d where $D = 10 - d$. Because of the pattern for dimensions $D \leq 5$, the groups $G^{(D)}$ are also often denoted as E_{d+1} in the literature.

As we saw for the ten-dimensional case in section 2.1, and will discuss further

Table 2.2: Classical symmetry groups $G(\mathbb{R})$ with maximal compact subgroups $K(\mathbb{R})$ and corresponding U-duality groups $G(\mathbb{Z})$ when compactifying on T^d to $D = 10 - d$ dimensions. Table is adapted from [34, 35] summarised in [18]. The split real forms $E_{n(n)}$ are here denoted E_n for brevity.

D	$G(\mathbb{R})$	$K(\mathbb{R})$	$G(\mathbb{Z})$
10	$SL_2(\mathbb{R})$	$SO_2(\mathbb{R})$	$SL_2(\mathbb{Z})$
9	$SL_2(\mathbb{R}) \times \mathbb{R}^+$	$SO_2(\mathbb{R})$	$SL_2(\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL_3(\mathbb{R}) \times SL_2(\mathbb{R})$	$SO_3(\mathbb{R}) \times SO_2(\mathbb{R})$	$SL_3(\mathbb{Z}) \times SL_2(\mathbb{Z})$
7	$SL_5(\mathbb{R})$	$SO_5(\mathbb{R})$	$SL_5(\mathbb{Z})$
6	$Spin_{5,5}(\mathbb{R})$	$\big(Spin_5(\mathbb{R}) \times Spin_5(\mathbb{R})\big)/\mathbb{Z}_2$	$Spin_{5,5}(\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp_8(\mathbb{R})/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU_8(\mathbb{R})/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin_{16}(\mathbb{R})/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

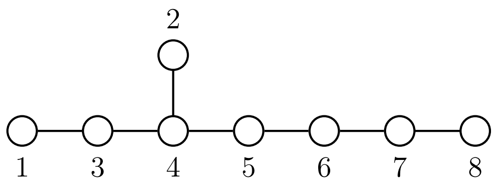


Figure 2.2: Dynkin diagram of E_8 with the Bourbaki convention for node labels. The groups of table 2.2 can be obtained by adding nodes in the order of the labeling.

below, the classical supergravity is invariant under $G(\mathbb{R})$-translations, but the quantum theory breaks this symmetry to the discrete Chevalley subgroup $G(\mathbb{Z})$ [29, 35, 73] which is defined in section 3.2. In Papers II and III we mainly work with SL_n for which the discrete Chevalley subgroup $SL_n(\mathbb{Z})$ coincides with the standard notion of integer matrices with determinant one.

This discrete symmetry is called U-duality (as in a unification of S- and T-dualities), and the group is shown in the third column in table 2.2. As mentioned in chapter 1, all physical observables are invariant under $G(\mathbb{Z})$-transformations of the moduli parameters.

We will now motivate why the U-duality group is the discrete Chevalley group $G(\mathbb{Z})$ from the view of the string theory in ten dimensions based on [3, 28, 59], and from the view of the field theory in four dimensions based on [35].

In section 2.1 we saw that the ten-dimensional type IIB supergravity theory (2.11) was invariant under the $SL_2(\mathbb{R})$ transformations (2.12). However, only a discrete subset of this classical symmetry can be a symmetry of the quantum theory (the type IIB string theory) which can be seen as follows. Consider a fundamental string carrying one unit-charge of the B-field (the 2-form B_2). From (2.12), we have that the B-field transforms as

$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \tag{2.23}$$

which means that the unit-charged string transforms to a string charged with d units of the B-field. However, in the quantum theory, the charges are quantised and d must therefore be an integer. The largest subgroup of $SL_2(\mathbb{R})$ where d is an integer is [59]

$$\left\{ \begin{pmatrix} a & \alpha b \\ \alpha/u & d \end{pmatrix} \middle| \, \alpha \in \mathbb{R},\, a,b,c,d \in \mathbb{Z},\, ad-bc=1 \right\} \tag{2.24}$$

but since α can be absorbed by a rescaling of C_2, this leaves $SL_2(\mathbb{Z})$.

It is conjectured that the discrete symmetry group of the quantum theory is, in fact, this largest allowed subgroup $SL_2(\mathbb{Z})$, something that cannot be proven directly but leads to a consistent picture of the web of M-theory dualities [59]. This discrete symmetry of the type IIB string theory is called an S-duality since it relates a strongly coupled theory to a weakly coupled theory. In particular, for $\chi_0 = 0$ and $\gamma = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$, we have that the constant mode $z = ig_\mathrm{s}^{-1}$ transforms to $\gamma(z) = -1/z = ig_\mathrm{s}$ meaning that $g_\mathrm{s} \to 1/g_\mathrm{s}$.

Following [29, 35], let us also study the $\mathcal{N} = 8$ supergravity theory in four dimensions, which is the low-energy effective action of type IIB (or IIA) string theory compactified on T^6. The particle content includes 28 vector bosons (or 1-forms) A^I with $I = 1, \ldots, 28$ whose fields strengths we denote by $F^I = dA^I$. Similar to the ten-dimensional case, the supergravity action is invariant under $G(\mathbb{R}) = E_7(\mathbb{R})$ transformations of the moduli fields and the field strengths. The transformations of the latter are easiest understood by taking a certain linear combination of the field strengths F^I and $*F^I$ forming a dual field strength G_I (in the sense of giving rise to electric Noether charges instead of magnetic charges) as detailed in [35]. They combine to a 56-dimensional field strength $\mathcal{F}$ attached to the vector representation of E_7 transforming as [35]

$$\mathcal{F} = \begin{pmatrix} F^I \\ G_I \end{pmatrix} \to \Lambda \begin{pmatrix} F^I \\ G_I \end{pmatrix} \qquad \Lambda \in E_7(\mathbb{R}) \subset Sp_{56}(\mathbb{R})\,. \tag{2.25}$$

The corresponding electric and magnetic Noether charges q_I and p^I are obtained by integrating over a two-sphere at spatial infinity

$$\mathcal{Q} = \begin{pmatrix} p^I \\ q_I \end{pmatrix} = \oint_{S^2} \mathcal{F} \tag{2.26}$$

which, together, are also attached to the vector representation of E_7 and satisfy the *Dirac–Schwinger–Zwanziger quantisation* condition for two particles (dyons) of charges $\mathcal{Q}$ and $\mathcal{Q}'$ [35]

$$^t\mathcal{Q}\Omega\mathcal{Q}' = \sum_I p^I q'_I - p'^I q_I \in \mathbb{Z} \tag{2.27}$$

where $^t\mathcal{Q}$ is the transpose of $\mathcal{Q}$ and

$$\Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \qquad \mathcal{Q} = \begin{pmatrix} p^I \\ q_I \end{pmatrix} \qquad \mathcal{Q}' = \begin{pmatrix} p'^I \\ q'_I \end{pmatrix}. \tag{2.28}$$

The quantisation condition (2.27) is invariant under $\mathcal{Q} \to S\mathcal{Q}$ and $\mathcal{Q}' \to S\mathcal{Q}'$ where $S \in Sp_{56}(\mathbb{R}) \supset E_7(\mathbb{R})$. However, if we assume that all types of $\mathcal{Q}$ charges exists, the only transformations S that preserve the lattice of charges are $S \in Sp_{56}(\mathbb{Z})$ [35]. Then, the U-duality group of the four-dimensional theory must be a subgroup of

$$E_7(\mathbb{R}) \cap Sp_{56}(\mathbb{Z}) \tag{2.29}$$

which was shown to coincide with the discrete Chevalley subgroup $E_7(\mathbb{Z})$ in [73]. Similar to the SL_2 case, it is conjectured the U-duality group is exactly $E_7(\mathbb{Z})$ [29,35].

For dimensions D larger than four, we note that the corresponding groups $G^{(D)}(\mathbb{R}) = G(\mathbb{R})$ in table 2.2 can be embedded in $E_7(\mathbb{R})$. It is then natural to conjecture that, for $D > 4$, the U-duality groups are [29, 35]

$$G^{(D)}(\mathbb{Z}) = G^{(D)}(\mathbb{R}) \cap E_7(\mathbb{Z})\,. \tag{2.30}$$

For dimensions D less than four, see [35].

The U-duality group for the type IIB string theory compactified on a torus T^d, or, equivalently, M-theory compactified on a torus T^{d+1}, can also be seen as a non-trivial intertwined product of the modular group $SL_{d+1}(\mathbb{Z})$ of T^{d+1} and the T-duality group as [3, 29]

$$G^{(10-d)}(\mathbb{Z}) = E_{d+1}(\mathbb{Z}) = SL_{d+1}(\mathbb{Z}) \bowtie SO(d,d;\mathbb{Z}) \tag{2.31}$$

which also contains the S-duality $SL_2(\mathbb{Z})$ discussed above surviving from the ten-dimensional theory.

2.4 Four-graviton scattering amplitudes

We will now consider the scattering amplitudes of four gravitons with momenta k_i^μ and polarisation tensors $\zeta_i^{\mu\nu}$ for $i = 1, \ldots, 4$. First, we will discuss the scattering amplitudes in ten-dimensional Minkowski space-time and, after that, give more general statements for toroidal compactifications.

Let t_8 be the standard rank eight tensor defined such that for an antisymmetric matrix $F_{\mu\nu}$

$$(t_8)^{\mu_1\nu_1\cdots\mu_4\nu_4} F_{\mu_1\nu_1} \cdots F_{\mu_4\nu_4} = 4 \operatorname{tr} F^4 - (\operatorname{tr} F^2)^2 \tag{2.32}$$

and let

$$\begin{aligned} K_{\text{cl}} &= \frac{1}{256} K_{\mu\nu\rho\sigma}(k_1, \ldots k_4) K_{\alpha\beta\gamma\delta}(k_1, \ldots k_4) \zeta_1^{\mu\alpha} \zeta_2^{\nu\beta} \zeta_3^{\rho\gamma} \zeta_4^{\sigma\delta} \\ K_{\mu\nu\rho\sigma}(k_1, \ldots, k_4) &= (t_8)_{\alpha\mu\beta\nu\gamma\rho\delta\sigma} k_1^\alpha k_2^\beta k_3^\gamma k_4^\delta\,. \end{aligned} \tag{2.33}$$

K_{cl} is the linearised form of [69, 74, 75]

$$t_8 t_8 R^4 = (t_8)^{\mu_1\rho_1\cdots\mu_4\rho_4} (t_8)^{\nu_1\sigma_1\cdots\nu_4\sigma_4} R_{\mu_1\rho_1\nu_1\sigma_1} \cdots R_{\mu_4\rho_4\nu_4\sigma_4}\,, \tag{2.34}$$

with $R_{\mu\nu\rho\sigma}$ being the Riemann curvature tensor. It is usual to denote $t_8 t_8 R^4$ simply by R^4 and K_{cl} as $\mathcal{R}^4$ where $\mathcal{R}$ is the linearised Riemann curvature tensor.

Define also the normalised Mandelstam variables

$$s = -\frac{\alpha'}{4}(k_1 + k_2)^2 \qquad t = -\frac{\alpha'}{4}(k_1 + k_3)^2 \qquad u = -\frac{\alpha'}{4}(k_1 + k_4)^2\,. \tag{2.35}$$

The first two leading-orders in g_s of the perturbative ten-dimensional four-graviton scattering amplitude [27, 60, 76] (reviewed in [74, 75]) are

$$\mathcal{A}_{\text{pert}} = g_\mathrm{s}^{-2} K_{\text{cl}} \Big(\mathcal{A}_{\text{tree}} + g_\mathrm{s}^2 \mathcal{A}_{\text{one-loop}} + \mathcal{O}(g_\mathrm{s}^3) \Big) \tag{2.36}$$

where the tree amplitude, whose computation is reviewed in section I-12.3, is

$$\mathcal{A}_{\text{tree}} = \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} \tag{2.37}$$

and the one-loop amplitude can be obtained as

$$\mathcal{A}_{\text{one-loop}} = 2\pi \int\limits_{\mathcal{F}} \frac{d^2\tau}{(\operatorname{Im}\tau)^2} \mathcal{B}_1(s,t,u|\tau)\,. \tag{2.38}$$

Here, $\mathcal{F}$ is fundamental domain of $SL_2(\mathbb{Z})$ on the upper half plane $\mathbb{H}$ parametrised by τ and $\mathcal{B}_1(s,t,u|\tau)$ is a modular invariant function in τ defined as [75]

$$\mathcal{B}_1(s,t,u|\tau) = \frac{1}{\tau_2^4} \prod_{i=1}^{4} \int\limits_{\Sigma_1(\tau)} d^2 z_i \exp\Big(\sum_{1\le i\le j\le 4} s_{ij} G(z_i - z_j|\tau) \Big) \tag{2.39}$$

where $\Sigma_1(\tau)$ is the torus with modulus τ over which the punctures z_i are integrated with $d^2z = \frac{i}{2} dz \wedge d\overline{z}$, $s_{12} = s_{34} = s$, $s_{13} = s_{24} = t$, and $s_{14} = s_{23} = u$. The function $G(z|\tau)$ is the *scalar Green's function* on $\Sigma_1(\tau)$ which is defined as

$$G(z|\tau) = -\log \left| \frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} \right|^2 - \frac{\pi}{2\tau}(z-\overline{z})^2 \tag{2.40}$$

where $\theta_1(z|\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi[(n+1/2)^2\tau + 2(n+1/2)(z+1/2)]}$ is the standard Jacobi theta-function. A similar computation for the vacuum to vacuum amplitude in the one-loop case is carried out in section I-12.4.

Remark 2.1. The integral over the fundamental domain $\mathcal{F}$ in the one-loop amplitude gives rise to branch cuts at zero for s, t and u, with the physical interpretation of infrared effects from massless strings in closed loops and give rise to non-local terms in the effective action [74, 75].

In order to get local interactions one must separate the analytical terms in (2.38) in a well-defined way. This can be done by splitting the integration domain into two parts above and below some large $\operatorname{Im}\tau = L$ and study the large L behaviour of the two terms. The lower integral captures the analytical terms together with terms diverging when $L \to \infty$. The latter cancel with similar terms in the upper integral which also contains the non-analytical terms [74, 75]. From here on, we will only consider the analytical terms. Non-analytic terms are treated, for example, in [77].

The Mandelstam variables depend on α', and using momentum conservation $s + t + u = 0$ we can make an α'-expansion of the scattering amplitude (2.36) in terms of

$$\sigma_2 = s^2 + t^2 + u^2 \qquad \sigma_3 = s^3 + t^3 + u^3 \tag{2.41}$$

as [74, 75]

$$\begin{aligned}\mathcal{A}_{\text{tree}} &= \frac{3}{\sigma_3} + 2\zeta(3) + \zeta(5)\sigma_2 + \frac{2}{3}\zeta(3)^2\sigma_3 + \mathcal{O}(\alpha'^{\,4}) \\ \mathcal{A}_{\text{one-loop}} &= 4\zeta(2) + \frac{4}{3}\zeta(2)\zeta(3)\sigma_3 + \mathcal{O}(\alpha'^{\,4})\,.\end{aligned} \tag{2.42}$$

This expansion in momenta, is also called a low-energy expansion.

The amplitude (2.36) is expressed in the string frame, but we will now convert to the Einstein frame, for which we recall the symmetries for the supergravity theory were most readily observed. If we gather the terms of (2.36) at each order in α', the amplitude can be expanded as [12]

$$\mathcal{A}_{\text{pert}} = K_{\text{cl}}\Big(\mathcal{E}_{(0,-1)}(z)\frac{1}{\sigma_3} + \sum_{p\geq 0}\sum_{q\geq 0}\mathcal{E}_{(p,q)}(z)\sigma_2^p\sigma_3^q\Big)\,. \tag{2.43}$$

The coefficients $\mathcal{E}_{(p,q)}$ are functions on the string theory moduli space $SL_2(\mathbb{R})/SO_2(\mathbb{R}) \cong \mathbb{H}$ parametrised by $z = x + iy = \chi_0 + ig_{\text{s}}^{-1}$, not to be confused with the moduli space of the world-sheet torus. The coefficient $\mathcal{E}_{(0,-1)}$ is constant and captures the corresponding supergravity amplitude [12].

From (2.42), converted to the Einstein frame, we have that

$$\begin{aligned}\mathcal{E}_{(0,0)}(z) &= 2\zeta(3)y^{3/2} + 4\zeta(2)y^{-1/2} + \ldots \\ \mathcal{E}_{(1,0)}(z) &= \zeta(5)y^{5/2} + \ldots \\ \mathcal{E}_{(0,1)}(z) &= \frac{2}{3}\zeta(3)^2y^3 + \frac{4}{3}\zeta(2)\zeta(3)y + \ldots\end{aligned} \tag{2.44}$$

Besides the contributions coming from string perturbation theory shown in (2.36), we will in section 2.4.2 see that the coefficients $\mathcal{E}_{(p,q)}$ also contain non-perturbative corrections in g_{s} of the form $\exp(-1/g_{\text{s}})$.

As discussed in section 2.3, all physical observables are invariant under U-duality transformations, which, for ten dimensions, are given by

$$z \to \gamma(z) = \frac{az+b}{cz+d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})\,. \tag{2.45}$$

This means that the coefficient functions $\mathcal{E}_{(p,q)}$ in the four-graviton scattering amplitude (2.43) are $SL_2(\mathbb{Z})$ invariant.

The four-graviton amplitude (or rather its analytic part) in D dimensions after compactifying on a torus takes a similar form as (2.43) with [12]

$$\mathcal{A}^{(D)}_{\text{pert}} = K_{\text{cl}}\Big(\mathcal{E}^{(D)}_{(0,-1)}(g)\frac{1}{\sigma_3} + \sum_{p\geq 0}\sum_{q\geq 0}\mathcal{E}^{(D)}_{(p,q)}(g)\sigma_2^p\sigma_3^q\Big) \tag{2.46}$$

where g is an element of the string theory moduli space $G(\mathbb{R})/K(\mathbb{R})$ shown in table 2.2 for different dimensions. We will continue to denote $\mathcal{E}^{(10)}_{(p,q)} = \mathcal{E}_{(p,q)}$ since the ten-dimensional case will often be used as an example.

The scattering amplitudes can be conveniently captured by a low-energy effective action, where momentum variables become derivates such that each coefficient $\mathcal{E}^{(D)}_{(p,q)}$ is accompanied by a derivative D^{2w} where $w = 2p + 3q$. Because of this, the α'-expansion is also called a derivative expansion. The part of the effective action that describes four-graviton interactions in Minkowski space-time is proportional to

$$S = \int d^D x \sqrt{-G}\Big(R + (\alpha')^3 \mathcal{E}^{(D)}_{(0,0)}(g) R^4 + (\alpha')^5 \mathcal{E}^{(D)}_{(1,0)}(g) D^4 R^4 + \\ + (\alpha')^6 \mathcal{E}^{(D)}_{(0,1)}(g) D^6 R^4 + \mathcal{O}(\alpha'^{\,7})\Big) . \quad (2.47)$$

where the first term is the Einstein-Hilbert term and where $R^4 = t_8 t_8 R^4$ from (2.34) is a contraction of four copies of the space-time Riemann curvature tensor.

Remark 2.2. Note that the scattering amplitude (2.46) depends on the constant mode $\chi_0 + i g_{\mathrm{s}}^{-1}$ of the axio-dilaton and other moduli fields, while the effective action depends on the whole field $\chi + i e^{-\phi}$. For convenience, we will use the same notation z for the full axio-dilaton and its constant mode when there is no risk of confusion.

The coefficients $\mathcal{E}^{(D)}_{(p,q)}$ are, similarly to the ten-dimensional case, invariant under the U-duality group $G(\mathbb{Z})$ shown in the third column of table 2.2. This invariance is one of the defining properties of an automorphic form which we define in section 3.3 in the adelic framework. Besides having well-behaved asymptotics, in for example the weak-coupling limit $g_{\mathrm{s}} \to 0$, which is here assured by the perturbative expansion (2.42), another defining property for an automorphic form $\mathcal{E}$ is that there exists a polynomial P such that $P(\Delta_{G/K})\mathcal{E} = 0$ where $\Delta_{G/K}$ is the Laplace-Beltrami operator on G/K. This is satisfied, for example, if $\mathcal{E}$ is an eigenfunction to $\Delta_{G/K}$. In the next section, we will see how supersymmetry gives such constraints for the leading-order coefficients $\mathcal{E}^{(D)}_{(p,q)}$.

2.4.1 Supersymmetry constraints

Again, let us start with the ten-dimensional case, and consider the space-time supersymmetry constraints for the leading-order correction $\mathcal{E}_{(0,0)} R^4$ based on [8, 32, 69, 78]. For more details, see section I-13.4.

The supersymmetry transformation relates different particles, or fields, with each other such as the dilaton with the dilatino. Thus, to obtain the supersymmetry constraints we are interested in, we need to take into account all the different interactions at the same order in α' that are related by supersymmetry. We label them as

$$S = S^{(0)} + (\alpha')^3 S^{(3)} + (\alpha')^5 S^{(5)} + \mathcal{O}(\alpha'^{\,6}) \quad (2.48)$$

where $S^{(0)} = S_{\mathrm{SUGRA}}$, $S^{(3)}$ contains the $\mathcal{E}_{(0,0)} R^4$ correction, and $S^{(5)}$ contains the $\mathcal{E}_{(1,0)} D^4 R^4$ correction.

The full supersymmetric action at order $(\alpha')^3$ has the form [8]

$$S^{(3)} = \int d^{10} x \sqrt{-G}\Big(f^{(12,-12)}(z)\lambda^{16} + f^{(11,-11)}(z)\hat{G}\lambda^{14} + \ldots \\ + f^{(0,0)}(z) R^4 + \ldots + f^{(-12,12)}(z)\lambda^{*16}\Big) \quad (2.49)$$

where we want to determine the functions $f^{(w,-w)}$. Note that we have briefly renamed $\mathcal{E}_{(0,0)}$ to $f^{(0,0)}$. The Weyl spinor field λ_α, $\alpha = 1\dots 16$, is the dilatino, the different powers of λ are defined as $(\lambda^n)_{\alpha_{n+1}\cdots\alpha_16} = \frac{1}{n!}\epsilon_{\alpha_1\cdots\alpha_16}\lambda^{\alpha_1}\cdots\lambda^{\alpha_n}$, ϵ is the Levi-Civita tensor, and $\hat{G}$ is a supercovariant[7] combination of the dilatino, gravitino and Ramond and Neveu-Schwarz two-form potentials which can be found in [8].

The interactions λ^{16}, $\hat{G}\lambda^{14}$ and λ^{*16}, for example, in (2.49) are not invariant under U-duality transformations but instead transform with some modular weights that needs to be absorbed by the transformations of the coefficients $f^{(w,-w)}(z)$ in order for the complete $S^{(3)}$ action to be invariant. The coefficients are thus (generalised) modular forms, where a modular form $f^{(w,\hat{w})}(z)$ of holomorphic weight w and anti-holomorphic weight $\hat{w}$ transforms as

$$f^{(w,\hat{w})}\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^w(c\bar{z}+d)^{\hat{w}} f^{(w,\hat{w})}(z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})\,. \tag{2.50}$$

The structure of the interactions in (2.49) and relations between the coefficients $f^{(w,-w)}$ can be obtained by studying the global limit of the linearised supersymmetry theory [8,32], where the supersymmetry parameter ϵ_α is taken to be constant and the supersymmetry transformations are considered to linear order in the field fluctuations. This is conveniently done in the superspace formalism discussed in section 2.1.

The superfield Φ describing the degrees of freedom for the ten-dimensional type IIB supergravity theory satisfies the holomorphicity constraint $D^*\Phi = 0$ where D^* is the covariant derivative from (2.15), and an on-shell condition $D^4\Phi = D^{*\,4}\Phi^* = 0$ [32,68]. As described in section 2.1, the space-time fields, or rather their linearised fluctuations, are obtained from the expansion of Φ in the Grassmann superspace variables θ and $\overline{\theta}$. Since we will, in this section, have to differentiate between the full axio-dilaton field and its constant mode we will write the full field as $z = z_0 + \delta z = x + iy$ where $z_0 = \chi_0 + i g_{\mathrm{s}}^{-1}$ is the constant mode and δz the fluctuations a round it.

The expansion is [8,32,69]

$$\begin{aligned}\Phi &= z_0 + \delta\Phi \\ &= z_0 + \delta z + \frac{1}{g_{\mathrm{s}}}\Big(i\overline{\theta}^*\delta\lambda + \delta\hat{G}_{\mu\nu\rho}\overline{\theta}^*\gamma^{\mu\nu\rho}\theta + \dots \\ &\qquad + \mathcal{R}_{\mu\sigma\nu\tau}\overline{\theta}^*\gamma^{\mu\nu\rho}\theta\overline{\theta}^*\gamma^{\sigma\tau}{}_\rho\theta + \dots + \theta^8\partial^4\overline{\delta z}\Big)\end{aligned} \tag{2.51}$$

where $\delta\Phi$ is the linearised fluctuation around the constant mode z_0 and a flat space-time metric, and $\mathcal{R}_{\mu\sigma\nu\tau}$ is the linearised Riemann curvature tensor.

Using Φ, a linearised version of $S^{(3)}$ can be obtained as an integral over half of superspace [8,32]

$$S^{(3)}_{\text{linear}} = \mathrm{Re}\int d^{10}x d^{16}\theta\theta F[\Phi] \tag{2.52}$$

where the integration $\int d^{16}\theta$ singles out the θ^{16} term in the expansion of the function $F[\Phi]$.

[7] Meaning that their supertransformations do not contain derivatives of the supersymmetry parameter ϵ_α.

That the integration is over half of superspace is connected to the fact that the D-instanton background that will be discussed in section 2.4.2 breaks 16 components of the supersymmetry transformation [32, 69].

Since the θ^4 term of the expansion (2.51) contains the linearised curvature tensor $\mathcal{R}_{\mu\nu\rho\sigma}$ the θ^{16} term will contain a contraction of four copies of $\mathcal{R}_{\mu\nu\rho\sigma}$. In fact, the superspace integral gives an alternative way of defining the $K_{\text{cl}} = \mathcal{R}^4$ factor as [69, 79]

$$\mathcal{R}^4 = \int d^{16}\theta (\mathcal{R}_{\mu\sigma\nu\tau}\overline{\theta}^* \gamma^{\mu\nu\rho}\theta \overline{\theta}^* \gamma^{\sigma\tau}{}_{\rho}\theta)^4 \,. \tag{2.53}$$

Taking the linearised approximation $y^{-1} \sim g_{\text{s}} \to 0$ of $S^{(3)}$ and comparing with the coefficients in the expansion of $F[\Phi]$ appearing in $S^{(3)}_{\text{linear}}$, we find that [32, 69]

$$f^{(12,-12)}(z) \sim y^{12}\Big(\frac{\partial}{\partial z}\Big)^{12} f^{(0,0)}(z) \tag{2.54}$$

up to a numerical factor that we will not need.

However, this relation is in contradiction with the fact that, while $f^{(0,0)}$ is invariant under $SL_2(\mathbb{Z})$ transformations, $f^{(12,-12)}$ should transform with holomorphic weight 12 and anti-holomorphic weight -12. This suggests that we should instead use a modular covariant derivative

$$D_{(w)} = i\Big(y\frac{\partial}{\partial z} - i\frac{w}{2}\Big) \tag{2.55}$$

which maps a modular form with weights $(w, \hat{w})$ to a modular form with weights $(w + 1, \hat{w} - 1)$. From (2.54), we would then have the relation

$$f^{(12,-12)}(z) \propto D_{(11)}D_{(10)} \cdots D_{(0)}f^{(0,0)} \,. \tag{2.56}$$

The other coefficients $f^{(w,-w)}$ are related in similar ways using $D_{(w)}$ and $\overline{D}_{(\hat{w})} \coloneqq \overline{D_{(\hat{w})}}$ the latter of which maps a modular for with weights $(w, \hat{w})$ to a modular form with weights $(w - 1, \hat{w} + 1)$.

Remark 2.3. The relation (2.56) approaches that of (2.54) in the weak coupling limit $y \to \infty$ for terms in $f^{(0,0)}$ that satisfy $y\partial_z f^{(0,0)}(z) \gg w f^{(0,0)}(z)$ as $y \to \infty$. After finding the solution for $f^{(0,0)}$ in section 2.4.3 we will see that this is satisfied by the non-perturbative contributions to $f^{(0,0)}$ meaning that the linearised approximation is an approximation for capturing the leading-order instanton contributions [8].

After having found the relation (2.56) we will now focus on the full non-linear theory again with local supersymmetry. The classical supergravity action $S^{(0)} = S_{\text{SUGRA}}$ from (2.11) is invariant under the classical supersymmetry transformations which we will denote by $\delta^{(0)}_\epsilon$ with the spinor ϵ being the supersymmetry transformation parameter, meaning that $\delta^{(0)}_\epsilon S^{(0)} = 0$. But as we add the higher-order corrections (2.48) to the action we also need to add corrections to the supersymmetry transformations

$$\delta_\epsilon = \delta^{(0)}_\epsilon + (\alpha')^3\delta^{(3)}_\epsilon + (\alpha')^5\delta^{(5)}_\epsilon + \dots \tag{2.57}$$

where, for example, the transformation $\delta_\epsilon^{(3)}$ (acting on some field) can be determined by an ansatz with coefficients to be solved for together with the coefficients $f^{(w,-w)}$ from $S^{(3)}$.

We focus on the transformation of λ^* and on terms with high powers of λ in the action $S^{(3)}$ to obtain a differential equation for $f^{(12,-12)}$ and, using (2.56), then obtain an equation also for $f^{(0,0)}$. By focusing on the dilatino it is possible to find differential equations that only involve $f^{(12,-12)}$, $f^{(11,-11)}$ and one unknown coefficient in $\delta^{(3)}\lambda^*$, which greatly simplifies the problem which, in general, would involve a larger number of coefficients mixing.

We now require that, to order $(\alpha')^3$, the corrected action (2.48) is invariant under the corrected supertransformations (2.57), and that the corrected supersymmetry algebra (or rather the full Poincaré algebra) closes up to equations of motion when acting on λ^*

$$\begin{aligned} \delta_{\epsilon_1}^{(0)} S^{(3)} + \delta_{\epsilon_1}^{(3)} S^{(0)} &= 0 \\ \delta_{\epsilon_2^*}^{(0)} S^{(3)} + \delta_{\epsilon_2^*}^{(3)} S^{(0)} &= 0 \\ [\delta_{\epsilon_1}, \delta_{\epsilon_2^*}]\lambda^* = \delta_{\text{local translation}}\lambda^* &+ \delta_{\text{local symmetries}}\lambda^* + (\text{equations of motion}) \end{aligned} \tag{2.58}$$

where $f^{(12,-12)}$ is present in $S^{(3)}$ as well as the equations of motion. The notation ϵ_1 and ϵ_2^* means that we consider supersymmetry transformation with different chiralities.

After some work (reviewed in section I-13.4), one finds that [8, 69]

$$4D_{(11)}\overline{D}_{(-12)} f^{(12,-12)}(z) = \Big(-132 + \frac{3}{4}\Big) f^{(12,-12)}(z) \tag{2.59}$$

which, using the relation (2.56), and replacing $f^{(0,0)}$ by $\mathcal{E}_{(0,0)}$, becomes

$$(\Delta_{\mathbb{H}} - \tfrac{3}{4})\mathcal{E}_{(0,0)}(z) = 0 \tag{2.60}$$

where $\Delta_{\mathbb{H}} = 4y^2 \partial_z \partial_{\overline{z}} = y^2(\partial_x^2 + \partial_y^2)$ is the Laplacian on the Poincaré upper half plane parametrised by $z = x + iy$. Thus, the R^4 coefficient $\mathcal{E}_{(0,0)}$ is an eigenfunction to the Laplacian, which was one of the defining properties of an automorphic form.

Note that if we can determine the coefficient $\mathcal{E}_{(0,0)}$, the covariant derivatives $D_{(w)}$ and $\overline{D}_{(\hat{w})}$ can be used to find all the other coefficients $f^{(w,-w)}$ in the action at this order in α' [69].

The method described above for the order $(\alpha')^3$ corrections was applied also to order $(\alpha')^5$ in [78] to obtain the following differential equation for the D^4R^4 coefficient

$$(\Delta_{\mathbb{H}} - \tfrac{15}{4})\mathcal{E}_{(1,0)}(z) = 0\,. \tag{2.61}$$

Using another method, it was shown in [21] however, that the D^6R^4 coefficient is not an eigenfunction to the Laplacian, but that it should instead satisfy the differential equation

$$(\Delta_{\mathbb{H}} - 12)\mathcal{E}_{(0,1)}(z) = -\big(\mathcal{E}_{(0,0)}(z)\big)^2\,. \tag{2.62}$$

For lower dimensions after toroidal compactifications the following similar differential equations were obtained in [12, 80, 81] for the R^4, D^4R^4 and D^6R^4 coefficients

$$\begin{aligned}
R^4: &\quad \Big(\Delta_{G/K} - \tfrac{3(11-D)(D-8)}{D-2}\Big)\mathcal{E}^{(D)}_{(0,0)}(g) = 6\pi\delta_{D,8} \\
D^4R^4: &\quad \Big(\Delta_{G/K} - \tfrac{5(12-D)(D-7)}{D-2}\Big)\mathcal{E}^{(D)}_{(1,0)}(g) = 40\zeta(2)\delta_{D,7} + 7\mathcal{E}^{(6)}_{(0,0)}(g)\delta_{D,6} \\
D^6R^4: &\quad \Big(\Delta_{G/K} - \tfrac{6(14-D)(D-6)}{D-2}\Big)\mathcal{E}^{(D)}_{(0,1)}(g) = \ -\big(\mathcal{E}^{(D)}_{(0,0)}(g)\big)^2 + 40\zeta(3)\delta_{D,6} + \\
&\qquad\qquad + \tfrac{55}{3}\mathcal{E}^{(5)}_{(0,0)}(g)\delta_{D,5} + \tfrac{85}{2\pi}\mathcal{E}^{(4)}_{(1,0)}(g)\delta_{D,4}
\end{aligned} \tag{2.63}$$

where $\Delta_{G/K}$ is the Laplace-Beltrami operator on the moduli space G/K.

The Kronecker delta sources on the right-hand-sides of the three equations in (2.63) are related to divergences in the supergravity theory, which are discussed in [36] in the case of the constant sources, and in [80, 81] for the remaining non-constant sources of the form $\mathcal{E}^{(n)}_{(p,q)}(g)\delta_{D,n}$.

For the coefficients $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(0,1)}$ we see that there exists polynomials P such that $P(\Delta_{G/K})\mathcal{E} = 0$ even for dimensions D giving non-zero sources. However, the third correction does not satisfy this condition, and, while invariant under $G(\mathbb{Z})$, it is thus strictly not an automorphic form.

Not only should the above constraint be satisfied by the Laplace-Beltrami operator, but, as we will see in the definition of an automorphic form in section 3.3, we require that there should exists similar polynomials for all G-invariant differential operators.

Such differential equations were obtained from supersymmetry constraints for the two leading-order corrections in various dimensions in [38, 39] using linearised harmonic superspace for maximal supersymmetry.

2.4.2 Instantons and non-perturbative effects

In this section we will first discuss what instantons are from a field theory point of view. We will then motivate why there should be instanton corrections to the four-graviton amplitude we obtained from string perturbation theory in the beginning of section 2.4 based on [82]. After that we will discuss instantons in type IIB supergravity and string theory to compute their contributions to the four-graviton scattering amplitude.

In field theory, instantons are, loosely speaking, non-trivial solutions to the equations of motion which give finite values for the action S. In the path integral formulation, the partition function Z, which is an integral of $\exp(-S)$ over all field configuration, can, using a semi-classical approximation, be approximated as a sum over local extrema to S multiplied by effects from quantum fluctuations around these extrema

$$Z \approx \sum_{\text{extrema}} e^{-S[\text{extremum}]} \times (\text{effects from quantum fluctuations around extremum})\,.$$

The global minimum is called the true vacuum around which the effects from quantum fluctuations give the ordinary perturbative corrections, and the local extrema correspond to corrections from instantons. When determining the effects of interactions in such a local extremum the computation is said to be made in an instanton background.

Let us now study instanton solutions in the ten-dimensional (Euclidean) type IIB supergravity theory. The following field configuration in spherical coordinates describes an instanton at the origin with Ramond-Ramond charge q [4, 83]

$$G_{\mu\nu} = \delta_{\mu\nu} \qquad e^{\phi} - e^{\phi_0} = \frac{|q|}{8\,\mathrm{Vol}(S^9)}\frac{1}{r^8} \qquad \chi - \chi_0 = \mathrm{sgn}(q)(e^{-\phi} - e^{-\phi_0}) \tag{2.64}$$

where $\phi_0 = \lim_{r\to\infty}\phi(r)$ such that $g_\mathrm{s} = e^{\phi_0}$, $\chi_0 = \lim_{r\to\infty}\chi(r)$ and $\mathrm{Vol}(S^9)$ is the volume of the unit nine-sphere. The charge q is the Noether-charge with respect to the translation symmetry in χ and is obtained by an integral over any hyper-surface enclosing the origin. The instanton is localised to a single point in space-time (the origin $r = 0$) in the sense that the charge q is invariant under deformations of the hyper-surface as long as the origin is not crossed. For positive q the solution is called an instanton, while for negative q it is called an anti-instanton. The charge becomes quantised according to the Dirac-Schwinger-Zwanziger quantisation condition as $q = 2\pi m$ where m is an integer.

Inserting the solution into the supergravity action one obtains the value [4, 83]

$$S_\mathrm{inst} = 2\pi\,|m|\,g_\mathrm{s}^{-1} - 2\pi i m\chi_0 = \begin{cases} -2\pi i\,|m|\,z_0 & \text{for instantons } (m > 0) \\ 2\pi i\,|m|\,\bar{z}_0 & \text{for anti-instantons } (m < 0) \end{cases} \tag{2.65}$$

where $z_0 = \chi_0 + i g_\mathrm{s}^{-1}$. This is the value of the action that enters as $S[\text{extremum}]$ in the approximation of the partition function above.

The string theory picture for the instanton corrections is a bit different from that of a field theory. So far we have, in section 2.2, only discussed how to compute perturbative corrections to the scattering amplitude using the g_s-expansion obtained from the diagrams of different genera shown in figure 2.1. Although a full understanding of non-perturbative string theory is yet to be developed, the existence of non-perturbative corrections was anticipated already in [82] based on the large genus behaviour the string perturbation theory where the amplitudes grow as $(2h)!$ for large genera h.

The divergent genus expansion should then be interpreted as an asymptotic series and Borel resummation techniques indicate the existence of non-perturbative terms of the form e^{-1/g_s} [82]. We note that a Taylor expansion of such a term around $g_\mathrm{s} = 0$ would vanish at all orders, meaning that these contributions would not be visible from string perturbation theory. They do however contribute with small values for finite couplings and dominate for strong couplings.

The string theory interpretation for the non-perturbative corrections anticipated in [82] come from D-branes and open strings. The diagrams in figure 2.1 for the genus expansion in string perturbation theory, used to compute the amplitude (2.36), included only closed strings, but when we add D-branes to the theory we also need to include open strings whose endpoints are attached to the D-branes. In particular, we will now study the contributions of D-instantons (which are localised to a single

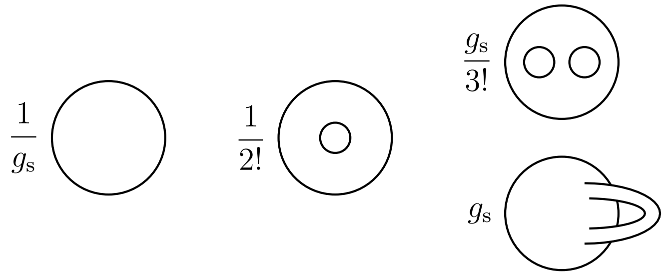


Figure 2.3: Leading-order open world-sheet components with g_s weights and symmetry factors for exchanging identical boundaries: a disk, an annulus, a disk with two holes, and a disk with a handle. All boundaries are attached to a single point in space-time.

point in space-time) to the four-graviton scattering amplitudes in ten dimensions following [84, 85].

With the inclusion of D-instantons in the theory the scattering amplitude becomes a sum over the number of D-instantons n at positions y_i, $i = 1, \ldots, n$ in space-time

$$\mathcal{A} = \sum_{n=0}^{\infty} \mathcal{A}_n \tag{2.66}$$

where $\mathcal{A}_0$ is the perturbative amplitude of (2.36). The amplitudes $\mathcal{A}_n$ only depend on the momenta and polarisations of the external states (as well as the string moduli) similar to $\mathcal{A}_0$ since the positions y_i are integrated over to restore translational invariance and momentum conservation for the external states.

When summing over possible world-sheet configurations for each $\mathcal{A}_n$ we will now allow for the world-sheets to have boundaries attached to the different D-instantons at positions y_i. Similar to ordinary Feynman diagrams in field theory, only connected diagrams give non-trivial contributions to the scattering amplitudes, but with the inclusion of D-instantons we can now allow the world-sheet surface Σ to be disconnected as long as the embedding in space-time is not, that is, as long as the boundaries of the world-sheet components of Σ are all connected together via the D-instantons [84].

Let us focus on the single D-instanton case $\mathcal{A}_1$. The world-sheet components are weighted by the Euler characteristic as $g_s^{2(h-1)+b}$ where h is the genus and b is the number of boundaries which, for $n = 1$, all are attached to the point y_1 of the D-instanton. The leading-order world-sheet components are: a disk, an annulus, a disk with two holes, and a disk with a handle as shown in figure 2.3 where we have also included symmetry factors for exchanging identical boundaries. External states are included as punctures on these world-sheets components.

For a given world-sheet configuration, each component will contribute with a factor to $\mathcal{A}_1$ from the world-sheet integral over the string-action (2.18). These factors will be denoted by $\langle \bigcirc \rangle$ for the disk and $\langle \circledcirc \rangle$ for the annulus, and so on. We will also separate the factor coming from components with punctures for the external states which we denote by $\mathcal{A}_1^{(\text{external})}$.

As for the closed string case, the amplitude $\mathcal{A}_1$ is a sum over all configurations, which here includes sums over the number of disks d_1 and the number of annuli d_2.

When including a symmetry factor for exchanging identical components (besides the symmetry factors in figure 2.3 for exchanging identical boundaries within such a component) we get an exponentiation in the amplitude [84, 85]

$$\begin{aligned}\mathcal{A}_1 &= \int d^{10}y_1 \sum_{d_1=0}^{\infty} \frac{1}{d_1!}\Big(\frac{1}{g_s}\langle\bigcirc\rangle\Big)^{d_1} \sum_{d_2=0}^{\infty} \frac{1}{d_2!}\Big(\frac{1}{2!}\langle\circledcirc\rangle\Big)^{d_2} \cdots \mathcal{A}_1^{(\text{external})} \\ &= \int d^{10}y_1 \exp\Big(\frac{1}{g_s}\langle\bigcirc\rangle + \frac{1}{2!}\langle\circledcirc\rangle + \ldots\Big)\mathcal{A}_1^{(\text{external})}\,.\end{aligned} \tag{2.67}$$

With a negative factor $\langle\bigcirc\rangle$, this gives us the expected non-perturbative correction of the form e^{-1/g_s}.

For the factor $\mathcal{A}_1^{(\text{external})}$ containing the punctures for the external states, the leading-order contributions come from four disks, each with one puncture, and give the same factor $\mathcal{R}^4$ as for the closed strings [4].

Including a non-zero axion, the following non-perturbative contribution to the R^4 coefficient $\mathcal{E}_{(0,0)}$ for the four-graviton scattering amplitude in ten dimensions was computed in [4]

$$\mathcal{E}_{(0,0)}(z) = 2\zeta(3)y^{3/2} + 4\zeta(2)y^{-1/2} + \ldots + Ce^{2\pi i z} + \ldots \tag{2.68}$$

where the perturbative terms come from (2.44) and C is an unkown constant. Note that this matches our expectations from both the large genus behaviour in string theory and the supergravity instanton action in field theory with a correction of the form $e^{-S_{\text{inst}}}$ for $m = 1$ in the instanton action (2.65).

We expect similar contributions for each instanton charge m where the prefactor C should contain a sum over the degeneracy of instanton states of charge m, each having the same value of the action S_{inst}. We will study this more closely in section 2.4.4, where we discuss the instanton measure, but we will first determine the degeneracy of D-instanton states in ten dimensions using T-duality.

Starting with a D-particle with integer Ramond-Ramond charge n in ten-dimensional type IIA string theory and compactifying the Euclidean time direction in space-time on a circle, the D-particle becomes a D-instanton in the T-dual limit where the radius of the compactified circle goes to zero. The T-dual action of such a D-particle with world-line wrapping the circle d times is [4]

$$S_{\text{inst}} = 2\pi|nd|y - 2\pi i n d x \tag{2.69}$$

which matches S_{inst} in (2.65) with the instanton charge $m = nd$ and $z_0 = \chi_0 + i g_s^{-1}$ replaced by $z = x + iy = \chi + ie^{-\phi}$. Thus, we can conclude that the degeneracy of instanton states of charge m is the number of divisors of m.

2.4.3 Leading-order coefficients in the derivative expansion

In this section we will discuss the U-duality invariant solutions to the differential equations (2.63) that satisfy the correct asymptotic behaviour in different limits, including, for example, the weak coupling limit $g_\text{s} \to \infty$ which can be studied by string perturbation theory.

We will start with the R^4 coefficient $\mathcal{E}_{(0,0)}(z)$ in the ten-dimensional theory, which we recall is $SL_2(\mathbb{Z})$ invariant, well-behaved in the weak coupling limit, and satisfy the eigenfunction equation (2.60) with eigenvalue $\frac{3}{4}$ and is thus an automorphic form.

Perhaps the most typical example of an automorphic form is a non-holomorphic Eisenstein series on $\mathbb{H} \cong SL_2(\mathbb{R})/SO_2(\mathbb{R})$ parametrised by $s \in \mathbb{C}$

$$E(s;z) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL_2(\mathbb{Z})} (\operatorname{Im} \gamma(z))^s = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{(\operatorname{Im} z)^s}{|cz+d|^{2s}} \tag{2.70}$$

where

$$\gamma(z) = \frac{az+b}{cz+d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{2.71}$$

and the Borel subgroup B is

$$B(\mathbb{R}) = \Big\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in SL_2(\mathbb{R}) \Big\} \qquad B(\mathbb{Z}) = \Big\{ \begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix} \in SL_2(\mathbb{Z}) \Big\}\,. \tag{2.72}$$

Note that these are not the holomorphic Eisenstein series usually discussed in the context of modular forms that transform with some modular weights.

To be able to generalise the above definition to higher-rank groups we will now rewrite the above Eisenstein series in group-theoretical terms. The group $SL_2(\mathbb{R})$ acts transitively on $\mathbb{H}$ by

$$\begin{gathered} SL_2(\mathbb{R}) \times \mathbb{H} \to \mathbb{H} \\ (g,z) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \mapsto g(z) = \frac{az+b}{cz+d} \end{gathered} \tag{2.73}$$

and since the stabiliser $\operatorname{Stab}(i) = SO_2(\mathbb{R})$, the map $gSO_2(\mathbb{R}) \mapsto g(i)$ gives a homeomorphism $SL_2(\mathbb{R})/SO_2(\mathbb{R}) \cong \mathbb{H}$. Using the Iwasawa decomposition $SL_2(\mathbb{R}) = N(\mathbb{R})A(\mathbb{R})K(\mathbb{R})$ where

$$N = \Big\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \Big\} \qquad A = \Big\{ \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \mid y \in \mathbb{R}_{>0} \Big\} \tag{2.74}$$

$$K = \Big\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi) \Big\} \tag{2.75}$$

any element $g \in SL_2(\mathbb{R})$ can be uniquely factorised as $g = nak$ with $n \in N(\mathbb{R})$, $a \in A(\mathbb{R})$ and $k \in K(\mathbb{R})$ where

$$g = nak = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \text{such that } g(i) = x + iy\,. \tag{2.76}$$

We introduce a multiplicative character χ (not to be confused with the axion χ) which is a map $B(\mathbb{Z})\backslash B(\mathbb{R}) \to \mathbb{C}^\times$ trivial on $N(\mathbb{R})$ that we trivially extend to $SL_2(\mathbb{R})$. In (2.70), the character is taken to be $\chi(g) = \operatorname{Im}(g(i))^s$. The Eisenstein series is then the sum over images of χ

$$E(\chi; g) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL_2(\mathbb{Z})} \chi(\gamma g) \,. \tag{2.77}$$

As we will see in section 3.4, we can also use a weight $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ to parametrise an Eisenstein series instead of a character χ.

The Eisenstein series are eigenfunctions to the Laplacian $\Delta_{\mathbb{H}} = 4y^2\partial_z\partial_{\bar{z}} = y^2(\partial_x^2 + \partial_y^2)$

$$\Delta_{\mathbb{H}} E(s; z) = s(s-1)E(s; z) \,. \tag{2.78}$$

They are also manifestly invariant under $SL_2(\mathbb{Z})$ transformations, and thus, in particular, periodic in $x = \operatorname{Re} z$. As detailed in appendix I-A, using Poisson resummation exploiting the lattice form of the right-hand-side of (2.70) and with the help of the fact that $SL_2(\mathbb{R})/SO_2(\mathbb{R}) \cong \mathbb{H}$, a Fourier expansion of the Eisenstein series can be computed as

$$E(s, z) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} + \frac{2y^{1/2}}{\xi(2s)} \sum_{m \in \mathbb{Z}\backslash\{0\}} |m|^{s-1/2} \sigma_{1-2s}(m) K_{s-1/2}(2\pi |m| y) e^{2\pi i m x} \tag{2.79}$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function, σ_s is the divisor sum

$$\sigma_s(n) = \sum_{\substack{d|n \\ d>0}} d^s \tag{2.80}$$

and $K_s(y)$ is the modified Bessel function of the second kind (which is exponentially decaying for large y).

Thus, we see that

$$E(s, z) \sim y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} \qquad y \to \infty \tag{2.81}$$

where we recall that the limit $y \to \infty$ corresponds to the weak coupling limit $g_{\mathrm{s}} \to 0$.

For $s = 3/2$, $E(s, z)$ satisfies the eigenfunction equation that is required for the R^4 coefficient $\mathcal{E}_{(0,0)}(z)$ and comparing the weak coupling behaviour of $\mathcal{E}_{(0,0)}$ in (2.44) with the asymptotic behaviour of $E(s, z)$ in (2.81) we make the following ansatz

$$\mathcal{E}_{(0,0)}(z) = 2\zeta(3)E(3/2, z) \,. \tag{2.82}$$

This ansatz agrees with all the terms from string perturbation theory, satisfies the supersymmetry constraints and, as will be shown in the next section, it also matches the expectation (2.68) for the instanton contributions discussed in section 2.4.2. In fact, it was shown in [9] that there are no additional automorphic forms (that is, cusp forms) with the right eigenvalue, that can be added to the ansatz without changing the weak coupling limit.

In a similar way for the D^4R^4 coefficient we can see that

$$\mathcal{E}_{(1,0)}(z) = \zeta(5)E(5/2, z)\,. \tag{2.83}$$

However, the D^6R^4 coefficient $\mathcal{E}_{(0,1)}$ is not an Eisenstein series, in fact, it is not strictly an automorphic form. A solution in terms of a Poincaré series, that is, a sum over images as in (2.77), but not of a character χ, was found in [31].

Solutions for the R^4 and D^4R^4 coefficients in dimensions $D \geq 3$ after compactifying on tori have been found as sums of (regularised) Eisenstein series in [12, 13, 18, 36]. In particular, they are, in several cases, proportional to maximal parabolic Eisenstein series defined in section 3.4. Indeed, with a maximal parabolic subgroup P_{α_1} defined in section 3.2 and $\lambda_s = 2s\Lambda_1 - \rho_P$ where ρ_P is defined in (3.32) we have that

$$\mathcal{E}^{(D)}_{(0,0)}(g) = 2\zeta(3)E_{P_{\alpha_1}}(\lambda_{s=3/2}, g) \tag{2.84}$$

for $3 \leq D \leq 7$, and

$$\mathcal{E}^{(D)}_{(1,0)}(g) = \zeta(5)E_{P_{\alpha_1}}(\lambda_{s=5/2}, g) \tag{2.85}$$

for $3 \leq D \leq 5$, where we recall that the differential equation (2.63) for $\mathcal{E}^{(D)}_{(1,0)}$ obtains Kronecker delta sources for $D = 6$ and $D = 7$.

Using (3.34) the maximal parabolic Eisenstein series can be expressed in terms of Eisenstein series with respect to the Borel subgroup.

As reviewed in [75], these solutions have been checked to agree with string perturbation theory in the weak coupling limit, and to be consistent in the decompactification limit, that is, taking the large radius limit of one of the compactified directions a coefficient in D dimensions recovers the coefficient in $D+1$ dimensions as will be discussed further in the next section. For ten dimensions, the R^4 coefficient was checked up to one-loop in [4], and the cancellation of the two-loop contribution shown in [86]. The tree-level and two-loop corrections to the D^4R^4 coefficient were checked in [30, 78, 87] and it was shown in [88] that the one-loop contribution vanishes. For the D^6R^4 coefficient the two-loop and three-loop contributions were verified in [89, 90]. For details about checks in lower dimensions, see [6, 7, 9, 10, 12, 13, 17, 20, 29, 80, 91].

The R^4 coefficient $\mathcal{E}^{(D)}_{(0,0)}$ can be shown to be attached to a minimal automorphic representation as defined in section 3.5 and the D^4R^4 coefficient $\mathcal{E}^{(D)}_{(1,0)}$ is attached to a next-to-minimal automorphic representation [13, 17, 18, 38]. As will be seen in section 3.7, this means that these functions have very few non-vanishing Fourier coefficients.

2.4.4 Physical information from Fourier coefficients

The coefficients $\mathcal{E}^{(D)}_{(p,q)}$ are invariant under $G(\mathbb{Z})$-transformations and may therefore be Fourier expanded with respect to different unipotent subgroups U of G as detailed in section 3.6 in the adelic framework.

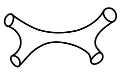
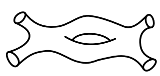
Depending on the choice of unipotent subgroup U, the Fourier coefficients carry information about different perturbative and non-perturbative effects.

Before treating lower dimensions, let us first consider the ten-dimensional case where there is only one choice of a unipotent subgroup U (inside the Borel subgroup) which amounts to expanding in $x = \operatorname{Re} z$ on $\mathbb{H}$. From the expansion of the SL_2 Eisenstein series in (2.79), we then already have a Fourier expansion of the R^4 and D^4R^4 coefficients $\mathcal{E}_{(0,0)}$ and $\mathcal{E}_{(1,0)}$. By expanding the Bessel function in (2.79) in the weak coupling limit $y \to \infty$ as

$$y^{1/2} K_{s-1/2}(2\pi\,|m|\,y) = e^{-2\pi|m|y}\left(\frac{1}{2\sqrt{|m|}} + \mathcal{O}(y^{-1})\right) \tag{2.86}$$

we find that

$$\mathcal{E}_{(0,0)}(z) = \overbrace{\underbrace{2\zeta(3)y^{3/2}}_{\text{tree-level}} + \underbrace{4\zeta(2)y^{-1/2}}_{\text{one-loop}}}^{\text{perturbative terms}} + \overbrace{2\pi\sum_{m\in\mathbb{Z}\setminus\{0\}}\sqrt{|m|}\sigma_{-2}(m)e^{-S_{\text{inst}}}\Big(1+\mathcal{O}(y^{-1})\Big)}^{\text{non-perturbative terms}}\,. \tag{2.87}$$

amplitudes in the presence of instantons

The instanton action in the exponential is the same as in (2.65)

$$S_{\text{inst}} = 2\pi\,|m|\,y^{-1} - 2\pi i m x\,. \tag{2.88}$$

and we see that expansion matches exactly the perturbative corrections from the genus expansion as well as the non-perturbative correction from the D-instanton computation (2.68). We also note that contributions from all instanton charges are present in (2.87) and that $e^{-S_{\text{inst}}}$ has a prefactor $\sigma_{-2}(m)$ which is a sum over the divisors of m. This factor is called the instanton measure denoted by $\mu(m)$, and, as anticipated in section 2.4.2, is a sum over the degeneracy of D-instanton states of charge m which are the different divisors of m. The expression $\mu(m) = \sigma_{-2}(m)$ was later proved in [92] using localisation techniques.

Also, there are only two perturbative terms in (2.87) which means that all the other genus diagrams do not contribute at this order in α'.

The expansion of $\mathcal{E}_{(1,0)}$ in ten dimensions have a similar physical interpretation, and we would now like to do the same analysis for a lower dimension D after compactifying on a torus T^d with $D = 10-d$. These Fourier coefficients are, however, very difficult to compute, especially if Poisson resummation techniques cannot be used (see section I-14.3 for more details), and methods for computing Fourier coefficients take a prominent rôle in the main objectives of this thesis.

We note that the non-trivial Fourier coefficient in (2.87) have an arithmetic factor consisting of the instanton measure, and an analytic part coming from the Bessel function. In the language of chapter 3, these are the p-adic part and the real part of the Fourier coefficient motivating the adelic framework that will be introduced in chapter 3 to compute these Fourier coefficients.

For higher-rank groups G, one can make several different Fourier expansions with respect to different unipotent subgroups U. We will discuss such Fourier expansions in detail in section 3.6, but will make some preliminary definitions here. See also chapter I-6.

We will mainly focus on unipotent subgroups U which are the unipotent radicals of maximal parabolic subgroups P as defined in section 3.2. The maximal parabolic subgroups are characterised by a choice of a simple root α, such that the Lie algebra of U is spanned by the positive Chevalley generators of roots with non-zero α-coefficient. A general definition can be found in section 3.2. These subgroups may not always be abelian, but we will here focus on the abelianisation $U^{\text{ab}} = [U,U]\backslash U$ where $[U,U] = \{u_1 u_2 u_1^{-1} u_2^{-1} \mid u_1, u_2 \in U\}$ is the commutator subgroup. See section 3.6 for more details.

Let $\mathfrak{u}$ and $\mathfrak{u}^{\text{ab}}$ be the Lie algebra of U and U^{ab} respectively, and let $\Delta(\mathfrak{u})$ be the roots associated to $\mathfrak{u}$, that is, the positive roots with non-zero α-coefficients. Then, the corresponding roots for $\mathfrak{u}^{\text{ab}}$ are $\Delta(\mathfrak{u}^{\text{ab}}) = \Delta(\mathfrak{u}) \setminus \Delta([\mathfrak{u},\mathfrak{u}])$ and we may parametrise an element $u \in U^{\text{ab}}(\mathbb{R})$ as

$$u = \prod_{\alpha \in \Delta(\mathfrak{u}^{\text{ab}})} \exp(u_\alpha E_\alpha) \qquad u_\alpha \in \mathbb{R} \tag{2.89}$$

where E_α is the Chevalley generator for the root α.

The Fourier modes are given by multiplicative characters ψ on $U(\mathbb{R})$ trivial on $U(\mathbb{Z})$ as well as $[U(\mathbb{R}), U(\mathbb{R})]$, which we can parametrise by integers m_α for $\alpha \in \Delta(\mathfrak{u}^{\text{ab}})$ as

$$\psi(u) = \exp\Big(2\pi i \sum_{\alpha \in \Delta(\mathfrak{u}^{\text{ab}})} m_\alpha u_\alpha\Big) \tag{2.90}$$

for u parametrised as above.

Then the Fourier coefficient of an automorphic form φ with respect to the character ψ on U is defined as

$$F_U(\varphi, \psi; g) = \int\limits_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varphi(ug)\psi^{-1}(u)\, du \tag{2.91}$$

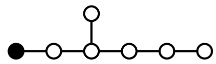
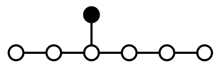
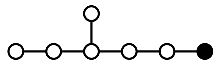
where the integration domain is such that we integrate over a period of the variables u_α and $\psi^{-1}(u) = \overline{\psi(u)}$. The Fourier coefficients are functions of the remaining variables in G, and in particular, of the Levi subgroup L of the maximal parabolic subgroup P which stabilises U under conjugation.

For a maximal parabolic subgroup P defined by a simple root α, there is a GL_1 factor in the Levi subgroup $L = GL_1 \times M$ of P which is related to the Chevalley generator H_α. We will parametrise this $GL_1(\mathbb{R})$ factor by a (suitably normalised) variable $r \in \mathbb{R}$ and study the asymptotic behaviour of the Fourier coefficients (2.91) when approaching the cusp $r \to 0$.

According to proposition I-13.6, we have, for a non-trivial character ψ and a spherical automorphic form φ, that is, an automorphic form that is right-invariant under the maximal compact subgroup $K(\mathbb{R})$ of $G(\mathbb{R})$, that

$$F_U(\varphi, \psi; g) \sim c_\psi(m) e^{-a_\psi/r} \exp\Big(2\pi i \sum_{\alpha \in \Delta(\mathfrak{u}^{\text{ab}})} m_\alpha u_\alpha\Big) \qquad \text{as } r \to 0 \tag{2.92}$$

Table 2.3: Different limits of string theory compactified on a torus T^d that can be studied by computing non-zero Fourier modes with respect to different parabolic subgroups P_α defined by the simple root α [12, 13, 17, 18]. The parabolic subgroups are illustrated in Dynkin diagrams for $D = 4$ where the unfilled nodes are part of the corresponding Levi subgroup.

String limit	Parabolic subgroup		Objects of study
String perturbation limit ($g_\mathrm{s} \to 0$)	P_{α_1}		D-instantons
M-theory limit (large M-theory torus)	P_{α_2}		M2- and M5-branes
Decompactification limit (large radius of compactified circle)	$P_{\alpha_{d+1}}$		higher-dimensional black holes

where $g \in G(\mathbb{R})$ can be parametrised by r, $m \in M(\mathbb{R})$, $u \in U(\mathbb{R})$ and some $k \in K(\mathbb{R})$ using the Langlands decomposition described in section 3.2, a_ψ is a constant and c_ψ a function on M. Note that the non-trivial Fourier modes, that is, the non-zero modes, are non-perturbative in r.

Thus, by computing Fourier coefficients with respect to different unipotent subgroups we can study different kinds of non-perturbative effects in string theory taking different variables r in the limit $r \to 0$. We will now list three such limits of importance in string theory based on [13, 17, 18] summarised in table 2.3.

First, we have the string perturbation limit where $g_\mathrm{s} \to 0$ which we have already studied in the ten-dimensional case. The maximal parabolic subgroup is defined by the simple root α_1 and the non-trivial Fourier modes carry information about D-instantons as in (2.87). The constant mode gives the perturbative corrections in g_s from string perturbation theory.

Then, the M-theory limit where we take the M-theory torus to be large. The M-theory torus is a $(d+1)$-torus T^{d+1} which is used in the compactification of the eleven-dimensional M-theory to obtain the type II string theory in $D = 10 - d$ dimensions. The maximal parabolic subgroup comes from the root α_2 and the non-trivial Fourier modes contain information about M2- and M5-brane states, while the constant mode captures the semi-classical approximation of the eleven-dimensional supergravity toroidally compactified to D dimensions.

Lastly, we have the decompactification limit where we take the radius of the circle for the extra compactified direction when going from $D+1$ to D dimensions to be large. The simple root for the corresponding maximal parabolic subgroup is α_{d+1}, which we note leaves the symmetry group G_{D+1} as part of the Levi subgroup, and the non-trivial Fourier modes carry information about higher dimensional states such as black hole BPS-states in $D+1$ dimensions whose world-lines wrap the extra circle. The constant modes of $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ with respect to the decompactification limit contain the coefficient functions $\mathcal{E}^{(D+1)}_{(0,0)}$ and $\mathcal{E}^{(D+1)}_{(1,0)}$ respectively for $D+1$ dimensions [13].

We will see in section 3.7 that the R^4, D^4R^4 and D^6R^4 coefficients receive corrections only from $\frac{1}{2}$-BPS, $\frac{1}{4}$-BPS and $\frac{1}{8}$-BPS states respectively by studying the vanishing properties of Fourier coefficients in the decompactification limit. In the superspace formalism in section 2.4.1, we have also seen that the R^4 correction is related to an integral over half of superspace.

We will now study the decompactification limit from four to five dimensions by first considering BPS-particles in five dimensions. As briefly discussed in section 2.4.1 and explained further in section I-13.3, the BPS fraction of preserved supersymmetry transformations for BPS-particles is determined by their electro-magnetic charges. Because of U-duality, these electro-magnetic charges are organised into BPS-orbits with all charges in the same orbit leading to the same BPS fraction.

In five dimensions, the BPS-particle charges are attached to a 27-dimensional vector representation of the group $G^{(5)} = E_6$ from table 2.2 and are organised into BPS-orbits under $GL_1 \times E_6$-transformations where the extra GL_1-factor is related to a scaling called a *trombone transformation* which is needed to generate the full $\frac{1}{8}$-BPS spectrum [93]. The orbits have dimensions 17, 26 and 27 for $\frac{1}{2}$-BPS, $\frac{1}{4}$-BPS and $\frac{1}{8}$-BPS respectively [18, 94].

These electro-magnetic charges and BPS-orbits correspond to the 27 integers m_α parametrising the character ψ on U for P_{α_7} in $D = 4$ where $G^{(4)} = E_7$, organised into character variety orbits (explained in section 3.6) under $L = GL_1 \times E_6$, where the GL_1-factor is here related to scalings of the extra compactified circle.

A similar story follows for all dimensions: For the Fourier coefficients of automorphic forms in D dimensions with respect to the decompactification limit, the integers parametrising the characters ψ are the electro-magnetic charges of BPS-particles in $D + 1$ dimensions. The electro-magnetic charges are organised into BPS-orbits which correspond to the character variety orbits.

To summarise, we have in this chapter, motivated from string theory why we want to compute Fourier coefficients of automorphic forms attached to small automorphic representations with respect to unipotent radicals of maximal parabolic subgroups. The leading-order coefficient functions $\mathcal{E}_{(p,q)}$ in the four-graviton scattering amplitude are automorphic forms attached to small automorphic representations and their Fourier coefficients with respect to different unipotent subgroups contain information about different kinds of non-perturbative effects in string theory. These Fourier coefficients are very difficult to compute, and are only known in certain cases.

In chapter 3 we will develop the theory of automorphic forms in the adelic framework, and in chapter 4 we will summarise the results of the appended papers to this thesis for computing their Fourier coefficients.

Chapter 3

Automorphic forms and representations

This chapter first introduces the ring of adeles and some Lie theory that will be used throughout the remaining part of the thesis. We will then define automorphic forms and representations on adelic groups, Eisenstein series, and Fourier coefficients of automorphic forms with respect to different unipotent subgroups.

As mentioned in section 1.2, we lift the coefficient functions $\mathcal{E}^{(D)}_{(p,q)}$ to adelic functions and then compute their adelic Fourier coefficients from which the real Fourier coefficients of interested in string theory can be obtained by a restriction of the argument.

Recommended references will mainly be listed at the beginning of the respective section. More details can also be found in Part 1 of Paper I.

3.1 The ring of adeles

This section introduces p-adic numbers and the ring of adeles based on [95–98].

The field of real numbers $\mathbb{R}$ is a Cauchy completion of the field of rational numbers $\mathbb{Q}$ with respect to the Euclidean norm. Instead of considering functions over $\mathbb{R}$, it is useful to take different completions of $\mathbb{Q}$, using different norms, and also to consider all of these completions at once.

Ostrowski's theorem [95, 99] tells us that the non-trivial norms on $\mathbb{Q}$ are either equivalent to the Euclidean norm or to a p-adic norm. For a prime p and a non-zero rational number q, which we prime-factorise as $q = p_1^{k_1} \cdots p_n^{k_n}$, we define the p-adic norm of q as

$$|q|_p = \begin{cases} p_i^{-k_i} & \text{if } p = p_i \text{ for any } i \\ 1 & \text{otherwise}\,, \end{cases} \tag{3.1}$$

and $|0|_p = 0$.

The completion of $\mathbb{Q}$ with respect to this norm is a field $\mathbb{Q}_p$ called the field of p-adic numbers. For convenience we will often use the notation $\mathbb{Q}_\infty$ for the real numbers $\mathbb{R}$ where $p = \infty$ is called the archimedean or real place and a prime $p < \infty$ is called a non-archimedean or finite place. The ring of integers of $\mathbb{Q}_p$, the p-adic integers denoted by $\mathbb{Z}_p$, are

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\} \supseteq \mathbb{Z} \tag{3.2}$$

and they are the p-adic completion of $\mathbb{Z}$.

A p-adic number $x \in \mathbb{Q}_p$ can be uniquely represented by a formal Laurent series

$$x = \sum_{i=k}^{\infty} x_i p^i \qquad k \in \mathbb{Z}, x_i \in \{0, 1, \ldots, p-1\} \text{ and } x_k \neq 0 \tag{3.3}$$

where k can be negative and where $|x|_p = p^{-k}$. The series is convergent in the p-adic norm and the partial sums form a Cauchy sequence for x connecting with the definition of $\mathbb{Q}_p$ as a Cauchy completion above.

We define the fractional part $[x]_p$ of a p-adic number x by its class in $\mathbb{Q}_p/\mathbb{Z}_p$ as

$$\Big[\sum_{i=k}^{\infty} x_i p^i\Big]_p = \begin{cases} \sum_{i=k}^{-1} x_i p^i & \text{if } k < 0 \\ 0 & \text{otherwise}. \end{cases} \tag{3.4}$$

For a rational number $q \in \mathbb{Q}$, we have, according to proposition I-2.13, that

$$q - \sum_{p \text{ prime}} [q]_p \in \mathbb{Z}. \tag{3.5}$$

The ring of adeles is defined as

$$\mathbb{A} = \mathbb{R} \times \prod_{p \text{ prime}}{}' \mathbb{Q}_p \tag{3.6}$$

where the prime on the product denotes that we restrict to elements $a = (a_\infty; a_2, a_3, a_5, a_7, \ldots)$ such that, for all but a finite number of primes p, $a_p \in \mathbb{Z}_p$. We also define the ideles $\mathbb{A}^\times$ as the group of units of $\mathbb{A}$, $\hat{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p$, and a global, adelic norm as

$$|a|_{\mathbb{A}} = \prod_{p \leq \infty} |a_p|_p \qquad a = (a_\infty; a_2, a_3, \ldots) \in \mathbb{A} \tag{3.7}$$

where the product is over all primes and the real place $p = \infty$. We will sometimes denote an element a as $(a_\infty; a_p)$ and suppress the subscript $\mathbb{A}$ for the norm.

The field $\mathbb{Q}$ is diagonally embedded in $\mathbb{A}$, is discrete in $\mathbb{A}$ (proposition I-2.24), and by factorising $q \in \mathbb{Q}^\times$ we see that $|q|_{\mathbb{A}} = 1$. The set $\{1\} \times \hat{\mathbb{Z}}$ is a subset of $\mathbb{A}$, which we will also denote by $\hat{\mathbb{Z}}$, and we have that $\mathbb{Q} \cap \hat{\mathbb{Z}} = \mathbb{Z}$ since $|q|_p \leq 1$ for all primes p means that the factorisation of q does not contain negative powers of any prime number p.

Let $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. We introduce a standard additive character on $\mathbb{A}$ trivial on $\mathbb{Q}$

$$\begin{aligned} \mathbf{e} : \mathbb{A} &\to U(1) \\ a &\mapsto e^{2\pi i a_\infty} \prod_{p<\infty} e^{-2\pi i [a_p]_p}. \end{aligned} \tag{3.8}$$

It can be shown that characters $\mathbf{e}(m\,\cdot\,)$ for $m \in \mathbb{Q}$ form all characters on $\mathbb{Q}\backslash\mathbb{A}$ [98]. Similar character will be used when we discuss Fourier expansions of automorphic forms in section 3.6.

In section I-2.2 we define integration on $\mathbb{Q}_p$ with respect to an additive measure dx invariant under translations $d(x+y) = dx$ and scaling as $d(yx) = |y|_p\,dx$ for $x, y \in \mathbb{Q}_p$ normalised such that the volume of $\mathbb{Z}_p$ is one. In the remaining part of section I-2.1 all the p-adic integrals appearing later in Paper I are computed as examples.

A function $f_\mathbb{A} : \mathbb{A} \to \mathbb{C}$ is called Eulerian if $f_\mathbb{A}(a) = \prod_{p\leq\infty} f_p(a_p)$ for some functions $f_p : \mathbb{Q}_p \to \mathbb{C}$. For an Eulerian function the adelic integral factorises as

$$\int_\mathbb{A} f(a)\,da = \int_\mathbb{R} f_\infty(a_\infty)da_\infty \prod_{p<\infty} \int_{\mathbb{Q}_p} f_p(a_p)\,da_p\,. \tag{3.9}$$

3.2 Lie theory and algebraic groups

The content of this section is summarised from [100–108]. More details can also be found in section I-3.1.1. We will, for simplicity, mainly focus on simple groups although much can be generalised for semisimple groups. Many of the Fourier coefficients of interest in string theory for the semisimple groups appearing in table 2.2 can be found in [18].

Let G be a finite-dimensional simple complex Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$, split real form $\mathfrak{g} = \mathfrak{g}_\mathbb{R}$ with corresponding real group $G(\mathbb{R})$, and a fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then, $\mathfrak{g}$ is decomposed as a sum of eigenspaces with respect to eigenvalues $\alpha : \mathfrak{h} \to \mathbb{R}$ in $\mathfrak{h}^*$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha \qquad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \quad \forall h \in \mathfrak{h}\} \tag{3.10}$$

where $\Delta \subset \mathfrak{h}^*$ is the set of non-zero α's (called *roots*) for which $\mathfrak{g}_\alpha \neq \{0\}$.

The space $\mathfrak{g}_\alpha$ is one-dimensional because of our assumptions on the Lie algebra, which means that, for each root α, there is a unique element $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ such that $\alpha(H_\alpha) = 2$.

We choose a consistent set of positive roots Δ_+ which is closed under addition (but not under subtraction) and contains exactly one of α and $-\alpha$ for each root. The set of simple roots Π is then the set of roots in Δ_+ which cannot be decomposed as a sum of two positive roots.

We denote the Killing form, which is a symmetric bilinear form mapping $\mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$, denoted by $\langle\cdot|\cdot\rangle$. It is compatible with the Lie bracket meaning that $\langle [x, y]|z\rangle = \langle x|[y, z]\rangle$.

Letting $T_\alpha = 2H_\alpha/\langle H_\alpha|H_\alpha\rangle$ we show in section I-3.1.1 that the Killing form defines an inner product on $\mathfrak{h}^*$ by how it acts on the roots $\alpha, \beta \in \Delta \subset \mathfrak{h}^*$

$$\langle\alpha|\beta\rangle = \langle T_\alpha|T_\beta\rangle = \alpha(T_\beta) = \beta(T_\alpha)\,. \tag{3.11}$$

We normalise the Killing form such that the unique highest root θ (with respect to the sum of its coefficients when expanded as a linear combination of simple roots) has length $\langle\theta|\theta\rangle$. For a weight $\lambda \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$ we will sometimes abuse notation and write $\langle\lambda|h\rangle$ for $\lambda(h)$.

For the simple roots α_i in Π where $i = 1, \ldots, r$ with r being the rank of $\mathfrak{g}$, we denote $H_i = H_{\alpha_i}$ and define fundamental weights $\Lambda_j \in \mathfrak{h}^*$ such that $\Lambda_j(H_i) = \delta_{ij}$. We also let ρ be the Weyl vector

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha = \sum_{i=1}^{r} \Lambda_i \,. \tag{3.12}$$

Parabolic subgroups Let B be the Borel subgroup of G such that its Lie algebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ where $\mathfrak{n} = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$. Standard parabolic subgroups are subgroups of G containing B. To construct a (standard) parabolic subgroup, choose a set Σ of simple roots which generate a subroot system $\langle\Sigma\rangle$. The parabolic subalgebra associated to Σ is then defined as

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{p})} \mathfrak{g}_\alpha \qquad \Delta(\mathfrak{p}) = \Delta_+ \cup \langle\Sigma\rangle \,. \tag{3.13}$$

This subalgebra has a Levi decomposition as $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{l}$ is a semisimple Levi subalgebra and $\mathfrak{u}$ the nilradical

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\langle\Sigma\rangle} \mathfrak{g}_\alpha \qquad \mathfrak{u} = \bigoplus_{\alpha\in\Delta_+\setminus(\Delta_+\cap\langle\Sigma\rangle)} \mathfrak{g}_\alpha \,. \tag{3.14}$$

Additionally, the Levi subalgebra has a Langlands decomposition into a semisimple and abelian part

$$\begin{aligned} \mathfrak{l} &= \mathfrak{m} \oplus \mathfrak{a}_P \\ \mathfrak{a}_P &= \{h \in \mathfrak{h} \mid \alpha(h) = 0 \quad \forall\alpha \in \Sigma\} \\ \mathfrak{m} &= [\mathfrak{l}, \mathfrak{l}] = \mathfrak{a}_P^\perp \oplus \bigoplus_{\alpha\in\langle\Sigma\rangle} \mathfrak{g}_\alpha \end{aligned} \tag{3.15}$$

where the orthogonal complement $\mathfrak{a}_P^\perp$ of $\mathfrak{a}_P$ in $\mathfrak{h}$ is taken with respect to the Killing form. We let P, L, M, A_P and U be the corresponding groups where $P = LU$ and $L = MA_P$ which factorise uniquely. The subgroup U is the unipotent radical of P.

In figure 3.1 we visualise the choice of simple roots $\Sigma = \{\alpha_2, \alpha_7\}$ for E_7 where the unfilled nodes represent the Levi subgroup of type $A_5 \times (GL_1)^2$ and the filled nodes represent the simple roots of $\mathfrak{g}$ whose Chevalley generators are in $\mathfrak{u}$.

If we choose $\Sigma = \emptyset$ then the parabolic subgroup becomes the Borel subgroup B whose Levi decomposition is denoted by $B = NA$ where N is the unipotent radical with Lie algebra $\mathfrak{n}$ from above and A is the maximal torus with respect to the $\mathfrak{h}$. An element of $G(\mathbb{R})$ can be uniquely factorised into the subgroups $N(\mathbb{R})A_0(\mathbb{R})K(\mathbb{R})$ called an Iwasawa decomposition where $A_0(\mathbb{R})$ is the connected component of $A(\mathbb{R})$ and $K(\mathbb{R})$ is a maximal compact subgroup [105, 109]. We will sometimes abuse notation and write $A(\mathbb{R})$ for $A_0(\mathbb{R})$ and call $A_0(\mathbb{R})$ the Cartan torus. For p-adic groups, that we will define below, the corresponding decomposition $G(\mathbb{Q}_p) = N(\mathbb{Q}_p)A(\mathbb{Q}_p)G(\mathbb{Z}_p)$

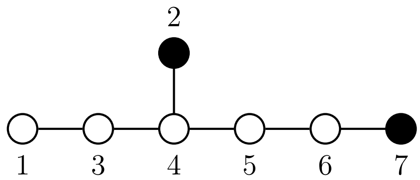


Figure 3.1: Visualisation of the parabolic subgroup corresponding to the choice of simple roots $\Sigma = \{\alpha_2, \alpha_7\}$.

is not uniquely factorisable, however the restriction to the norm on $A(\mathbb{Q}_p)$ is, which we will use when defining Eisenstein series below. For all other standard parabolic subgroups we have a non-unique factorisation $G = PK$ for both $\mathbb{R}$ and $\mathbb{Q}_p$.

If we choose $\Sigma = \Pi \setminus \{\alpha_i\}$ for some simple root α_i, the parabolic subgroup, which we denote P_{α_i}, is called a maximal parabolic subgroup. As seen in table 2.3 they are especially important in string theory.

Algebraic groups We will now define p-adic and adelic groups, as well as the discrete Chevalley groups discussed in section 2.3, following [109–113]. In the beginning of this section we considered complex Lie groups and their split real forms. More generally, one can define an algebraic group G over a field k (of characteristic 0) as follows.

An algebraic variety X over k can roughly be thought of as a set of polynomial equations with coefficients in k and, for a commutative k-algebra R, the R-points of X are the solutions of these equations in R. The functor taking k-algebras to sets of solutions, by mapping R to the R-points of X, determines the variety X up to isomorphisms [111, 114].

An algebraic group G is an algebraic variety over k with regular maps that act as the ordinary group multiplication and inversion, as well as a multiplicative identity.

Let R be a commutative k-algebra, V a vector space over k, and let GL_V be the functor $R \rightsquigarrow \mathrm{Aut}_R(R \otimes_k V)$ which is an algebraic group for finite-dimensional V, where $\mathrm{Aut}_R(R \otimes_k V)$ is the group of R-linear automorphisms on $R \otimes_k V$.

A linear algebraic group is an algebraic group which has a faithful finite-dimensional representation (ρ, V). This gives an isomorphism between G and an algebraic subgroup of GL_V (given by further polynomial equations with coefficients in k). If we have a linear algebraic group over $\mathbb{Q}$, we can then take R as the $\mathbb{Q}$-algebras $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}_p$ or $\mathbb{A}$ to obtain the group $G(R)$ considered as a subgroup of $GL_V(R)$.

Remark 3.1. For convenience, we will, for the remaining part of this section, mainly consider algebraically simply connected, reductive (e.g. semisimple), linear algebraic groups as defined in [110, 111].

For each finite-dimensional complex simple Lie algebra $\mathfrak{g}$ we will show how to obtain such a linear algebraic group over $k_0 = \mathbb{Q}$, called a Chevalley group, by constructing generators parametrised by values in $K = \mathbb{C}$ [113, 115]. The $\mathbb{C}$-points of the algebraic Chevalley group gives the usual complex simple Lie group corresponding to $\mathfrak{g}$.

The same process can be used for any algebraically closed field K and its prime subfield[8] k_0 to construct a linear algebraic group over k_0 of Lie type $\mathfrak{g}$.

To be able to obtain corresponding groups for $\mathbb{Z}$ and $\mathbb{Z}_p$, we will take this construction further and define the Chevalley–Demazure group scheme $G_\mathbb{Z}$ over $\mathbb{Z}$ which can be seen as a functor taking any commutative ring R with a unit to a group $G(R)$ [115,116]. As will be explained further below, when R is an algebraically closed field K, $G_\mathbb{Z}(K)$ gives the Chevalley group constructed from K.

Let us start by constructing the Chevalley group over $\mathbb{Q}$. For a finite-dimensional simple complex Lie algebra $\mathfrak{g}$ a construction by Chevalley and Steinberg gives a semisimple linear algebraic group G defined and split over $\mathbb{Q}$ as follows [73, 113, 115, 117]. Let P and Q be the weight lattice and the root lattice respectively

$$\begin{aligned} P &= \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_r \\ Q &= \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_r \end{aligned} \tag{3.16}$$

and choose a lattice L such that $Q \subseteq L \subseteq P$.

Let (ρ, V) be a faithful representation of $\mathfrak{g}$, with V a finite-dimensional complex vector space, such that L is the lattice of weights of ρ (such a representation exists according to [113]), and let $V_\mathbb{Z}$ be a $\mathbb{Z}$-lattice of V invariant under $\rho(E_\alpha)^n/n!$ for all roots $\alpha \in \Delta$ of $\mathfrak{g}$ and integers $n > 0$. Such a lattice $V_\mathbb{Z}$ is called admissible.

For $t \in \mathbb{C}$ and $\alpha \in \Delta$ introduce

$$x_\alpha(t) = \exp(t\rho(E_\alpha)) = \sum_{n=0}^{\infty} t^n \frac{\rho(E_\alpha)^n}{n!} \tag{3.17}$$

where we note that $\rho(E_\alpha)$ is nilpotent [73].

The map x_α is a rational homomorphism from the additive group of $\mathbb{C}$ into GL_V, and, since the image of an algebraic group under a rational homomorphism is an algebraic group, $X_\alpha = \{x_\alpha(t) : t \in \mathbb{C}\}$ is an algebraic subgroup of GL_V [113]. We define the Chevalley group G to be the algebraic subgroup of GL_V generated by X_α for all $\alpha \in \Delta$. G can be shown to only depend on the root system of $\mathfrak{g}$ and the choice of lattice L [113,115]. We will therefore often omit writing the representation (ρ, V).

If we choose L to be the weight lattice P, then G is simply connected and called the universal Chevalley group of type $\mathfrak{g}$, and if we choose L to be the root lattice Q, then G is the adjoint Chevalley group. We will, in this thesis, mainly consider the simply connected groups, which are listed in table 3.1. The groups PSL_{n+1} and SO_{2n+1} are examples of adjoint Chevalley groups, and SO_{2n} is an example of a Chevalley group that is obtained by a lattice $Q \subsetneq L \subsetneq P$.

It can also be shown that G is an algebraic group over $\mathbb{Q}$ instead of $\mathbb{C}$, that is, using the basis given by the admissible lattice $V_\mathbb{Z}$, the polynomial equations have coefficients in $\mathbb{Q}$ [113]. This means that we now can obtain the R-points $G(\mathbb{Q})$, $G(\mathbb{R})$, $G(\mathbb{Q}_p)$ and $G(\mathbb{A})$, but to be able to define the corresponding groups for $\mathbb{Z}$ and $\mathbb{Z}_p$ we note the following.

Since the coefficients in the polynomial equations are in $\mathbb{Q}$ we may multiply by the denominators to obtain polynomial equations with integer coefficients which then

[8] That is, the unique subfield of K which does not contain any proper subfields.

Table 3.1: Simply connected Chevalley groups for different types of Lie algebras $\mathfrak{g}$ obtained by the choice of lattice $L = P$, with P being the weight lattice. The table is adapted from [113].

Type of $\mathfrak{g}$	G (simply connected)
A_n	SL_{n+1}
B_n	$Spin_{2n+1}$
C_n	Sp_{2l}
D_n	$Spin_{2n}$
E_6	$E_{6,\text{sc}}$
E_7	$E_{7,\text{sc}}$
E_8	E_8
F_4	F_4
G_2	G_2

generate a subring $\mathbb{Z}[G]$ of the coordinate ring of G. The Chevalley–Demazure group scheme $G_{\mathbb{Z}}$ over $\mathbb{Z}$ is then defined by the functor of points [118–120]

$$R \rightsquigarrow G_{\mathbb{Z}}(R) = \mathrm{Hom}(\mathbb{Z}[G], R) \tag{3.18}$$

which takes a commutative ring R with 1 to a group $G_{\mathbb{Z}}(R)$ (which we can think of as the R-solutions to the above polynomial equations with integer coefficients).

We then define the group $G(\mathbb{Z})$ to be the $\mathbb{Z}$-points of $G_{\mathbb{Z}}$ and similarly for $\mathbb{Z}_p$. Explicitly, we have that [73, 109]

$$\begin{aligned} G(\mathbb{Z}) &= \{g \in G(\mathbb{Q}) \mid g(V_{\mathbb{Z}}) = V_{\mathbb{Z}}\} \\ G(\mathbb{Z}_p) &= \{g \in G(\mathbb{Q}_p) \mid g(V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p) = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p\} \end{aligned} \tag{3.19}$$

which compares with the discussion in section 2.3 where $G(\mathbb{Z})$ are the elements preserving the charge lattice.

Thus, we have well-defined groups $G(R) = G_{\mathbb{Z}}(R)$ for $R = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{Z}$, $\mathbb{Z}_p$ and $\mathbb{A}$ for any simple Lie type, and we will now find convenient parametrisations for them using a similar construction as for the complex Chevalley groups above.

For a commutative ring R with 1, the group generated by $x_\alpha(t)$ for all $\alpha \in \Delta$ and $t \in R$ is a subgroup of $G_{\mathbb{Z}}(R)$ (sometimes a proper subgroup) [119] and is called the elementary Chevalley group denoted by $E(R)$.

If R is an algebraically closed field K, then $G(K) = E(K)$ for any choice of lattice $Q \subseteq L \subseteq P$ [118, 119], that is, the Chevalley construction from above works equally well for any algebraically closed field K as for $\mathbb{C}$. When R is a local ring, for example $\mathbb{Z}_p$ or an arbitrary field and $L = P$, meaning that G is algebraically simply connected, then $G(R) = E(R)$ according to [118, Proposition 1.6]. As a consequence of [117, Theorem 2.3], this is also true for $R = \mathbb{Z}$.

Thus, we have convenient parametrisations for $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{Z}$ and $\mathbb{Z}_p$, and thus also for $\mathbb{A}$ since the adelic group $G(\mathbb{A})$ factorises as [109, 110]

$$G(\mathbb{A}) = G(\mathbb{R}) \times G_f \qquad G_f = \prod_{p<\infty}{}' G(\mathbb{Q}_p) \tag{3.20}$$

where the primed product again signifies that the all but a finite number of factors are in $G(\mathbb{Z}_p)$.

Furthermore, for a field k we can, for $t \in k^\times$, also define

$$\begin{aligned} w_\alpha(t) &= x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t) \\ h_\alpha(t) &= w_\alpha(t)w_\alpha(1)^{-1} \end{aligned} \tag{3.21}$$

which are related to the Weyl group and the Cartan torus with respect to the Chevalley basis. Indeed, in the defining representation of SL_2 these are

$$w_\alpha(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad h_\alpha(t) = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \tag{3.22}$$

The generators h_α for all $\alpha \in \Pi$ commute and are multiplicative as functions of t_α, and for any field k, an element in the Cartan torus of $E(k)$ can be parametrised by [113]

$$h = \prod_{\alpha\in\Pi} h_\alpha(t_\alpha) \qquad t_\alpha \in k^\times \tag{3.23}$$

which then also parametrise the Cartan tori of $G(\mathbb{Q})$, $G(\mathbb{R})$, $G(\mathbb{Q}_p)$ and, by extension, $G(\mathbb{A})$.

Strong approximation Let G be a simply connected, simple linear algebraic group over $\mathbb{Q}$. The group $G(\mathbb{Q})$ is diagonally embedded in $G(\mathbb{A})$ similar to $\mathbb{Q}$ in $\mathbb{A}$ and is discrete in $G(\mathbb{A})$ [109].

If $G(\mathbb{R})$ is non-compact (which is the case for the split real form), then $G(\mathbb{Q})$ is dense in $G(\hat{\mathbb{Z}}) = K_f = \prod_{p<\infty} G(\mathbb{Z}_p)$ which is called the strong approximation property [109, 121].

Theorem I-3.9. *If $G(\mathbb{Q})$ is dense in $G_f = \prod'_{p<\infty} G(\mathbb{Q}_p)$, K_Γ an open subgroup of K_f and $\Gamma = K_\Gamma \cap G(\mathbb{Q})$, then*

$$\begin{aligned} \phi : \Gamma\backslash G(\mathbb{R}) &\to G(\mathbb{Q})\backslash G(\mathbb{A})/K_\Gamma \\ \Gamma x_\infty &\mapsto G(\mathbb{Q})(x_\infty; \mathbb{1})K_\Gamma \end{aligned} \tag{3.24}$$

is a homeomorphism.

Note that K_Γ is embedded in $G(\mathbb{A})$ as $(\mathbb{1}; k_p)$. We will often use $K_\Gamma = K_f$ giving $\Gamma = G(\mathbb{Z})$ and we may then use ϕ to lift a function $f : G(\mathbb{Z})\backslash G(\mathbb{R}) \to \mathbb{C}$ to a function $F = f \circ \phi^{-1} : G(\mathbb{Q})\backslash G(\mathbb{A})/K_f \to \mathbb{C}$ which is called an adelic lift. If we consider F and f as functions on $G(\mathbb{A})$ and $G(\mathbb{R})$ respectively, we have that $F\big((x; \mathbb{1})\big) = f\big(\phi^{-1}\big((x; \mathbb{1})\big)\big) = f(x)$ for $x \in \mathbb{R}$.

If f is also spherical, meaning that it is a function on $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$ where $K(\mathbb{R})$ is the maximal compact subgroup shown in table 2.2, we have that $F = f \circ \phi^{-1}$: $G(\mathbb{Q})\backslash G(\mathbb{A})/K_\mathbb{A}$ where $K_\mathbb{A} = K(\mathbb{R}) \times K_f$. This is the case for the $G(\mathbb{Z})$-invariant (U-duality invariant) functions $\mathcal{E}^{(D)}_{(p,q)}$ on the moduli space $G(\mathbb{R})/K(\mathbb{R})$ that were part of the scattering amplitudes we considered in chapter 2.

Remark 3.2. The strong approximation gives a uniqueness to the adelisation of a function $f : G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$. If $\tilde{F} : G(\mathbb{Q})\backslash G(\mathbb{A})/K_{\mathbb{A}}$, seen as a function on $G(\mathbb{A})$, is such that $\tilde{F}((x;\mathbb{1})) = f(x)$ for all $x \in G(\mathbb{R})$, then $\tilde{F} \circ \phi = f$, which means that $\tilde{F} = F = f \circ \phi^{-1}$.

We will see that lifting the string theory coefficients $\mathcal{E}^{(D)}_{(p,q)}$ to the adeles will enable us to compute their Fourier coefficients which carry information about non-perturbative effects in string theory as seen in section 2.4.4.

Remark 3.3. In section 3.6 we will also use the following homeomorphism for a unipotent subgroup U of G which can be shown in a similar way as theorem I-3.9 (see remark I-3.12 for more details). If $G(\mathbb{Q})$ is dense in G_f and $U(\hat{\mathbb{Z}}) = \prod_{p<\infty} U(\mathbb{Z}_p)$ is an open subgroup of G_f, then

$$\begin{aligned} \Phi : U(\mathbb{Z})\backslash U(\mathbb{R}) &\to U(\mathbb{Q})\backslash U(\mathbb{A})/U(\hat{\mathbb{Z}}) \\ U(\mathbb{Z})x_\infty &\mapsto U(\mathbb{Q})(x_\infty;\mathbb{1})U(\hat{\mathbb{Z}}) \end{aligned} \tag{3.25}$$

is a homeomorphism.

3.3 Definition of automorphic forms

This section is based on [110, 112, 122, 123]. See also chapter I-4 for more details.

In chapter 2 we briefly mentioned the defining properties of an automorphic form on real groups. Since we can lift these functions to the adeles (something that will make later computations much easier), we will here give a detailed definition of an automorphic form on adelic groups. The definition for real groups is similar and can be seen in section I-1.1.

Definition 3.1. *An (adelic) automorphic form φ is a smooth function $G(\mathbb{A}) \to \mathbb{C}$ that satisfies the following four conditions for $g \in G(\mathbb{A})$:*

1. **automorphic invariance:** $\varphi(\gamma g) = \varphi(g) \quad \forall \gamma \in G(\mathbb{Q})$
2. **K-finiteness:** $\dim_{\mathbb{C}}(\mathrm{span}\{\varphi(gk) \mid k \in K_{\mathbb{A}}\}) < \infty$
3. **Z-finiteness:** $\dim_{\mathbb{C}}(\mathrm{span}\{D_X\varphi(g) \mid X \in \mathcal{Z}(\mathfrak{g}_{\mathbb{R}})\}) < \infty$
4. **moderate growth:** *For any norm $\|\cdot\|$ on $G(\mathbb{A})$ there exists a positive integer n and a constant C such that $|\varphi(g)| \leq C\|g\|^n$.*

Here, $K_{\mathbb{A}} = K(\mathbb{R}) \times K_f = K(\mathbb{R}) \times \prod_{p<\infty} G(\mathbb{Z}_p)$ and $\mathcal{Z}(\mathfrak{g}_{\mathbb{R}})$ is the centre of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{R}})$ acting on φ as bi-invariant differential operators D_X. An example of such a differential operator is the Casimir operator.

Because of the form of the string moduli spaces, we will mostly consider spherical automorphic forms, that is, automorphic forms that are right-invariant under $K_{\mathbb{A}}$-translations, meaning that they automatically satisfy condition 2. Section I-4.2 provides an example of non-spherical automorphic forms, where we show how a classical modular form on the upper half plane can be described as an automorphic form on $SL_2(\mathbb{R})$ which transforms by a phase under the right action of $K(\mathbb{R})$. The holomorphicity condition for the modular form becomes an eigenvalue equation for the Laplacian with eigenvalue depending on the modular weight.

Condition 3 is equivalent to the requirement that if $X \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$, then there exists a polynomial P such that $P(X)\varphi = 0$. Note that this requirement is satisfied by the R^4 and D^4R^4 coefficients $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ in the four-graviton amplitude because of the supersymmetry constraints as shown in section 2.4.1.

The last condition means that φ should be well-behaved in different limits (or cusps). For example, $\mathcal{E}^{(D)}_{(p,q)}$ should be well-behaved in the weak coupling limit or decompactification limit as discussed in section 2.4.4.

The space of automorphic forms φ satisfying these conditions is denoted by $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$.

3.4 Eisenstein series

A typical example of an automorphic form is an Eisenstein series. In section 2.4.3 we introduced non-holomorphic Eisenstein series on $SL_2(\mathbb{R})$ as a sum over $B(\mathbb{Z})\backslash SL_2(\mathbb{Z})$ where B is the Borel group (2.72).

We will now generalise this for any group $G(\mathbb{A})$. For higher-rank groups we can define Eisenstein series with respect to any parabolic subgroup P, not only the Borel subgroup. Let P be the parabolic subgroup defined by the set of simple roots Σ as seen in section 3.2. As also described there, we have a unique factorisation $P(\mathbb{A}) = M(\mathbb{A})A_P(\mathbb{A})U(\mathbb{A})$ but a non-unique factorisation $G(\mathbb{A}) = P(\mathbb{A})K_\mathbb{A}$ where $K_\mathbb{A} = K(\mathbb{R}) \times K_f$.

However, $A_P(\mathbb{A}) \subset A(\mathbb{A})$ and the restriction of an element $g \in G(\mathbb{A})$ to $A(\mathbb{A})$ is, because of the Iwasawa decomposition, well-defined for $A(\mathbb{R})$ and also for $A(\mathbb{Q}_p)$ after taking the p-adic norm as detailed below.

As in (3.23), let $a \in A(\mathbb{A})$ be parametrised as

$$a = \prod_{\alpha\in\Pi} h_\alpha(t_\alpha) \qquad t_\alpha \in \mathbb{A}^\times \tag{3.26}$$

where t_α is not uniquely defined, but $|t_\alpha|_\mathbb{A}$ is. For a weight $\lambda \in \mathfrak{h}^*_\mathbb{C}$ we define a map $\mathfrak{h}^*_\mathbb{C} \times A(\mathbb{A}) \to \mathbb{C}$

$$(\lambda, a) \mapsto |a|^\lambda := \prod_{\alpha\in\Pi} |t_\alpha|_\mathbb{A}^{\langle\lambda|H_\alpha\rangle} \tag{3.27}$$

which satisfies

$$|a|^{\lambda_1}|a|^{\lambda_2} = |a|^{\lambda_1+\lambda_2} \qquad (|a|^{\lambda_1})^s = |a|^{(s\lambda_1)} \qquad |a_1|^\lambda |a_2|^\lambda = |a_1a_2|^\lambda \tag{3.28}$$

for $a, a_i \in A(\mathbb{A})$, $\lambda_i \in \mathfrak{h}^*_\mathbb{C}$ and $s \in \mathbb{C}$. Since $|q|_\mathbb{A} = 1$ for $q \in \mathbb{Q}$, we have that $|a|^\lambda = 1$ for $a \in A(\mathbb{Q})$. We make the analogous definitions for $|a|_p^\lambda$ using the p-adic norm.

Using the above map, we may then define the logarithm map

$$\begin{aligned} H : A(\mathbb{A}) &\to \mathfrak{h}_\mathbb{R} \\ a &\mapsto \sum_{\alpha\in\Pi} H_\alpha \log|a|^{\Lambda_\alpha} \end{aligned} \tag{3.29}$$

such that

$$H\big(\prod_{\beta\in\Pi} h_\beta(t_\beta)\big) = \sum_{\alpha\in\Pi} H_\alpha \log|t_\alpha|_{\mathbb{A}} \quad \text{and} \quad |a|^\lambda = e^{\langle\lambda|H(a)\rangle}\,. \tag{3.30}$$

We will mainly use $|a|^\lambda$ in favor of $e^{\langle\lambda|H(a)\rangle}$ although both are heavily used in the literature.

As seen from the decompositions in (3.14) and (3.15), L stabilises U and A_P stabilises M under conjugation. Let $p_1, p_2 \in P(\mathbb{A})$ factorising as $p_i = l_i u_i$ and $l_i = m_i a_i$. The product $p_1 p_2$ factorises as $p_1 p_2 = (l_1 l_2)\,(l_2^{-1} u_1 l_2 u_2)$ with the first parenthesis in L and the second in U, and similarly for $l_1 l_2 = m_1 a_1 m_2 a_2 = (m_1 a_1 m_2 a_1^{-1})\,(a_1 a_2)$.

Thus, the map $\chi : P(\mathbb{Q})\backslash P(\mathbb{A}) \to \mathbb{C}^\times$ defined by

$$\chi(p) = \chi(m a_P u) = |a_P|^{\lambda+\rho_P} \tag{3.31}$$

for some $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ is a multiplicative character where p is uniquely factorised as $m a_P u$ with $m \in M(\mathbb{A})$, $a_P \in A_P(\mathbb{A})$, $u \in U(\mathbb{A})$, and ρ_P is defined as

$$\rho_P = \frac{1}{2} \sum_{\alpha\in\Delta_+\backslash(\Delta_+\cap\langle\Sigma\rangle)} \alpha \tag{3.32}$$

which coincides with the usual Weyl vector (3.12) for $P = B$ when $\Sigma = \emptyset$.

As described in section I-5.7, the character χ_P only depends on the projection of $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ onto $\mathfrak{a}^*_P = \operatorname{span}_{\mathbb{C}}\{\Lambda_\alpha \mid \alpha \in \Pi\backslash\Sigma\}$. Since the character, by definition, depends only on the norm on $A_P(\mathbb{A})$, its trivial extension to $G(\mathbb{A})$ is well-defined.

We define the (spherical) Eisenstein series with respect to P and χ as

$$E_P(\chi; g) = \sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \chi(\gamma g)\,. \tag{3.33}$$

We will often interchange χ and λ in $E_P(\chi; g) = E_P(\lambda; g)$ using (3.31).

These are the Eisenstein series that occur in (2.84) and (2.85) in the string scattering amplitudes with P being a maximal parabolic subgroup. Other typical Eisenstein series are those defined for the minimal parabolic where $P = B$ is the Borel subgroup, which are usually denoted simply by $E(\chi; g)$ without a subscript B. For a maximal parabolic subgroup P_α defined by $\Sigma = \Pi \setminus \{\alpha\}$ as described in section 3.2, and for $\lambda = 2s\Lambda_\alpha - \rho$ we show in proposition I-5.28 and (I-5.98) that

$$E(\lambda; g) = E_{P_\alpha}(\lambda; g) = E_{P_\alpha}(\lambda_P; g)\,, \tag{3.34}$$

where $\lambda_P = 2s\Lambda_\alpha - \rho_P$. This is why, we for the rest of this thesis, will focus on minimal parabolic Eisenstein series on the Borel subgroup B.

By restricting the argument to $g = (g_\infty; \mathbb{1}, \mathbb{1}\ldots)$ we recover the standard Eisenstein series defined over $G(\mathbb{R})$ [13] which are straight-forward generalisation of the $SL_2(\mathbb{R})$ Eisenstein series given in section 2.4.3.

From appendix I-B we have that

$$\Delta_{G/K} E(\lambda; g) = \frac{\langle\lambda|\lambda\rangle - \langle\rho|\rho\rangle}{2} E(\lambda; g)\,. \tag{3.35}$$

As reviewed in section I-8.8, Langlands [124] showed that the Eisenstein series satisfy the functional equation

$$E(\lambda; g) = M(w,\lambda)E(w\lambda; g) \qquad M(w,\lambda) = \prod_{\substack{\alpha\in\Delta_+\\ w\alpha<0}} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(1+\langle\lambda|\alpha\rangle)} \tag{3.36}$$

for each Weyl element $w \in \mathcal{W}$. The completed Riemann zeta function $\xi(s)$ for $s \in \mathbb{C}$ was defined after (2.79) as $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

As also discussed in section I-8.8, the Eisenstein series defined in (3.33) converges when λ is in the Godemont range defined by

$$\mathrm{Re}\langle\lambda|\alpha\rangle > 1 \qquad \forall\alpha \in \Pi\,. \tag{3.37}$$

Using the functional relation above, the Eisenstein series can then be analytically continued to almost all $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$.

3.5 Automorphic representations

This section introduces automorphic representations following [122,123,125]. Consider the space $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ of automorphic forms on $G(\mathbb{A})$. The group $G_f = \prod'_{p<\infty} G(\mathbb{Q}_p)$ acts on this space by the right-regular action

$$[\pi(g_f)\varphi](h) = \varphi(hg_f) \qquad h \in G(\mathbb{A}), g_f \in G_f \tag{3.38}$$

for $\varphi \in \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. A similar right-regular action for elements $g_\infty \in G(\mathbb{R})$ does not preserve the K-finiteness condition, and thus takes us outside the space of automorphic forms [123]. By definition, the maximal compact subgroup $K(\mathbb{R})$ does however preserve the K-finiteness condition under right-translations.

Besides the right-translations by G_f and $K(\mathbb{R})$, the space of automorphic forms also carries an action by the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\mathbb{R})$ as differential operators D_X for $X \in \mathcal{U}(\mathfrak{g}_\mathbb{R})$. The actions by $K(\mathbb{R})$ and $\mathcal{U}(\mathfrak{g}_\mathbb{R})$ both commute with the action by G_f, but not with each other. Instead they give $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ the structure of a $(\mathfrak{g}_\mathbb{R}, K(\mathbb{R}))$-module which is defined as a vector space with a K-finiteness condition as in section 3.3 and with $\mathfrak{g}_\mathbb{R}$ and $K(\mathbb{R})$ actions that satisfy

$$\begin{aligned} \pi(k_\infty)D_X\,\varphi &= D_{k_\infty^{-1}Xk_\infty}\pi(k_\infty)\,\varphi && k_\infty \in K(\mathbb{R}), X \in \mathcal{U}(\mathfrak{g}_\mathbb{R}) \\ \frac{d}{dt}\Big(\pi(e^{tY})\varphi\Big)\Big|_{t=0} &= D_Y\varphi && Y \in \mathrm{Lie}(K(\mathbb{R}))\,. \end{aligned} \tag{3.39}$$

Definition 3.2. *An automorphic representation (π, V) of $G(\mathbb{A})$ is an irreducible component V of $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ under the simultaneous action by $(\mathfrak{g}_\mathbb{R}, K(\mathbb{R})) \times G_f$ as described above.*

We denote the subspace of V attached to an irreducible representation σ of $K(\mathbb{R})\times K_f$ under right-translations as $V[\sigma]$ and call a representation (π, V) admissible if $V[\sigma]$ is finite-dimensional for all σ. It was shown in [126] that an admissible automorphic representation (π, V) factorises as

$$(\pi, V) = \bigotimes_{p \leq \infty} (\pi_p, V_p) \tag{3.40}$$

where (π_∞, V_∞) is a $(\mathfrak{g}_\mathbb{R}, K(\mathbb{R}))$-module, and, for finite p, (π_p, V_p) is a representation of $G(\mathbb{Q}_p)$.

If, for such an automorphic representation, a local representation (π_p, V_p) for finite p contains a non-zero vector f_p invariant under $G(\mathbb{Z}_p)$, π_p is called unramified, or spherical, and f_p is called a spherical vector. If π_p is spherical for almost all p, the global representation (π, V) is called an unramified, or spherical, automorphic representation. We will for the rest of this thesis mainly consider spherical automorphic representations.

Let χ be a multiplicative character on $B(\mathbb{A})$ and consider the following subspace of smooth functions on $G(\mathbb{A})$

$$\operatorname{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} \chi = \{ f : G(\mathbb{A}) \to \mathbb{C} \mid f(bg) = \chi(b) f(g) \quad \forall b \in B(\mathbb{A}), g \in G(\mathbb{A}) \} \tag{3.41}$$

(with natural generalisations for other subgroups) which is an induced representation of $G(\mathbb{A})$ from the one-dimensional representation of $B(\mathbb{A})$ given by χ. The induced representation is called the principal series. From any section $f_\chi \in \operatorname{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} \chi$ we can define an Eisenstein series by summing over the images of f_χ similar to section 3.4 where f_χ was χ itself. We can thus see the Eisenstein series as a map

$$\begin{aligned} E : \operatorname{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} &\to \mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A})) \\ f_\chi &\mapsto \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} f_\chi(\gamma g) \,. \end{aligned} \tag{3.42}$$

The notation $\operatorname{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} \chi$ will be used in chapter 4. There we will also discuss minimal and next-to-minimal automorphic representations which will be defined in section 3.7.

3.6 Fourier coefficients

Fourier coefficients of automorphic forms on $G(\mathbb{R})$ were briefly discussed in section 2.4.4. We will now study the corresponding adelic objects.

Let φ be an automorphic form on $G(\mathbb{A})$ and $U(\mathbb{A})$ be a unipotent subgroup of $G(\mathbb{A})$, not to be confused with $U(1) = \{ z \in \mathbb{C} \mid |z| = 1 \}$. A unitary multiplicative character $\psi : U(\mathbb{Q}) \backslash U(\mathbb{A}) \to U(1)$ is a function on $U(\mathbb{A})$ trivial on $U(\mathbb{Q})$ and satisfy $\psi(u_1 u_2) = \psi(u_1)\psi(u_2)$ for $u_i \in U(\mathbb{A})$. This means that ψ is rather a function on the abelianisation $U^{\text{ab}} = [U, U] \backslash U$ where $[U, U] = \{ u_1 u_2 u_1^{-1} u_2^{-1} \mid u_1, u_2 \in U \}$ is the commutator subgroup.

The Fourier coefficient of φ on U with respect to the character ψ is

$$F_U(\varphi, \psi; g) = \int\limits_{U(\mathbb{Q}) \backslash U(\mathbb{A})} \varphi(ug) \psi^{-1}(u) \, du \tag{3.43}$$

where $\psi^{-1}(u) = \overline{\psi(u)}$. We will sometimes use the short-hand notation $[U] = U(\mathbb{Q})\backslash U(\mathbb{A})$.

By a change of variables, we see that $F_U(\varphi, \psi; u'g) = \psi(u')F_U(\varphi, \psi; g)$ for $u' \in U(\mathbb{A})$ since ψ is multiplicative. This means that the Fourier coefficients $F_U(\varphi, \psi; g)$ cannot capture the behaviour of φ on the commutator subgroup $[U, U]$. Therefore, to recover the automorphic form ϕ, a complete Fourier expansion must include Fourier coefficients on $[U, U]$ as well if U is not abelian, and so on for further commutator subgroups.

Let $U^{(i+1)} = [U^{(i)}, U^{(i)}]$ where $U^{(0)} = U$, and $\psi^{(i)}$ a unitary multiplicative character on $U^{(i)}(\mathbb{Q})\backslash U^{(i)}(\mathbb{A})$. Then, the automorphic form φ can be Fourier expanded as

$$\varphi(g) = F_{U^{(0)}}(\varphi, 1; g) + \sum_{\psi^{(0)} \neq 1} F_{U^{(0)}}(\varphi, \psi^{(0)}; g) + \sum_{\psi^{(1)} \neq 1} F_{U^{(1)}}(\varphi, \psi^{(1)}; g) + \cdots \tag{3.44}$$

The procedure terminates after a finite number of steps since U is unipotent. The first term is called the constant term with respect to U.

When U is the unipotent radical of a parabolic subgroup P, we call the Fourier coefficient a parabolic Fourier coefficient. As seen in section 2.4.4, we are, in particular, interested in maximal parabolic Fourier coefficient.

Furthermore, when $U = N$, that is, the unipotent radical of the Borel subgroup B, the corresponding Fourier coefficients are called Whittaker coefficients denoted by $W_N(\varphi, \psi; g) = F_N(\varphi, \psi; g)$ and have been extensively studied in the literature. We have collected and developed methods for computing them as seen in section 4.1. In section 4.2, we have computed Fourier coefficients of other parabolic subgroups in terms of the known Whittaker coefficients.

For a spherical automorphic form φ, invariant under right-translations by $K_{\mathbb{A}}$, such as the spherical Eisenstein series we defined in section 3.4, we find that $W_N(\varphi, \psi; g)$ is determined by its values on $A(\mathbb{A})$ where $G(\mathbb{A}) = N(\mathbb{A})A(\mathbb{A})K(\mathbb{A})$ is the Iwasawa decomposition

$$W_N(\varphi, \psi; nak) = W_N(\varphi, \psi; na) = \psi(n)W_N(\varphi, \psi; a)\,. \tag{3.45}$$

In section 4.1 we will therefore only have to determine the Whittaker coefficients as functions on $A(\mathbb{A})$.

As seen above, the character ψ is effectively a function on the abelianisation $U^{\text{ab}} = [U, U]\backslash U$. Let $\Delta(\mathfrak{u})$ and $\Delta(\mathfrak{u}^{\text{ab}}) = \Delta(\mathfrak{u}) \setminus \Delta([\mathfrak{u}, \mathfrak{u}])$ be the roots of the corresponding Lie algebras. Then, an element $u \in U^{\text{ab}}(\mathbb{A})$ can be parametrised by

$$u = \prod_{\alpha \in \Delta(u^{\text{ab}})} x_\alpha(u_\alpha) \qquad u_\alpha \in \mathbb{A} \tag{3.46}$$

where $x_\alpha(t) = \exp(tE_\alpha)$ from (3.17). As shown in proposition I-6.9, and already discussed in section 2.4.4, a character ψ on $U(\mathbb{Q})\backslash U(\mathbb{A})$ can be parametrised by a set of rational number m_α for $\alpha \in \Delta(\mathfrak{u}^{\text{ab}})$ as

$$\psi(u) = \mathbf{e}\Big(\sum_{\alpha \in \Delta(\mathfrak{u}^{\text{ab}})} m_\alpha u_\alpha\Big) \tag{3.47}$$

where $\mathbf{e}$ is the character on $\mathbb{Q}\backslash\mathbb{A}$ from (3.8). From the definition of $\mathbf{e}$ we see that ψ factorises as

$$\begin{gathered}\psi(u) = \psi_\infty(u_\infty)\prod_{p<\infty}\psi_p(u_p) \qquad u = (u_\infty; u_2, u_3, \ldots) \in U(\mathbb{A}) \\ \psi_\infty(u_\infty) = \exp\Big(2\pi i \sum_{\alpha\in\Delta(\mathfrak{u}^{\mathrm{ab}})} m_\alpha u_{\alpha,\infty}\Big) \qquad \psi_p(u_p) = \exp\Big(-2\pi i \sum_{\alpha\in\Delta(\mathfrak{u}^{\mathrm{ab}})} [m_\alpha u_{\alpha,p}]_p\Big)\end{gathered} \tag{3.48}$$

where ψ_∞ and ψ_p are unitary multiplicative characters on $U(\mathbb{Z})\backslash U(\mathbb{R})$ and $U(\mathbb{Z}_p)\backslash U(\mathbb{Q}_p)$ respectively. As explained in Paper III, the charges m_α can also be seen as parametrising an element in a conjugated space of the Lie algebra $\mathfrak{u}$.

When all $m_\alpha = 0$ the character becomes trivial $\psi = 1$. If all m_α are non-zero, then the character (and the corresponding Fourier coefficient) is called generic, and otherwise it is called degenerate. We will in chapter 4 often encounter maximally degenerate character which only have one non-zero m_α.

Because of the physical interpretation discussed in section 2.4.4 we often call the rational numbers m_α instanton charges.

We mentioned in section 3.4 that the Eisenstein series (3.33) on $G(\mathbb{A})$ reduced to the Eisenstein series on $G(\mathbb{R})$ when restricting $g \in G(\mathbb{A})$ to $g = (g_\infty; \mathbb{1}, \mathbb{1}, \ldots)$ with $g_\infty \in G(\mathbb{R})$ and analogously for adelic lifts of other functions. Using strong approximation, we will now see that a similar restriction for a Fourier coefficient with respect to $U(\mathbb{A})$ reduces to a Fourier coefficient with respect to $U(\mathbb{R})$ defined in (2.91).

By remark 3.3, we have that $\Phi : U(\mathbb{Z})\backslash U(\mathbb{R}) \to U(\mathbb{Q})\backslash U(\mathbb{A})/U(\hat{\mathbb{Z}})$ defined in (3.25) is a homeomorphism. Recall that the corresponding map for $G(\mathbb{A})$ is denoted by ϕ. Let $\psi_\infty : U(\mathbb{Z})\backslash U(\mathbb{R}) \to U(1)$ be a unitary multiplicative character. As mentioned in section 2.4.4, ψ is determined by integer charges m_α according to (2.90). Now use the same integer charges m_α for an adelic character $\psi : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1)$ from (3.47). Then, for $k = \prod_{\alpha\in\Delta(\mathfrak{u}^{\mathrm{ab}})} x_\alpha(k_\alpha) \in U(\hat{\mathbb{Z}})$ with $k_\alpha \in \hat{\mathbb{Z}}$,

$$\psi(k) = 1 \cdot \prod_{p<\infty} \exp\Big(-2\pi i \sum_{\alpha\in\Delta(\mathfrak{u}^{\mathrm{ab}})} [m_\alpha k_{\alpha,p}]_p\Big) = 1 \tag{3.49}$$

since $m_\alpha k_{\alpha,p} \in \mathbb{Z}_p$ for integer m_α which implies that $[m_\alpha k_{\alpha,p}]_p = 0$. We also have that, for all $(u_\infty; \mathbb{1}) \in U(\mathbb{A})$,

$$\psi((u_\infty; \mathbb{1})) = \psi_\infty(u_\infty) \prod_{p<\infty} \psi_p(\mathbb{1}) = \psi_\infty(u_\infty) \tag{3.50}$$

meaning that ψ is the unique adelisation of ψ_∞ using a similar statement as the one in remark 3.2.

Proposition I-6.20. *Let ψ_∞ and ψ be as above, and let $f : G(\mathbb{Z})\backslash G(\mathbb{R}) \to \mathbb{C}$ with the adelisation $\varphi = f \circ \phi^{-1} : G(\mathbb{Q})\backslash G(\mathbb{A})/K_f \to \mathbb{C}$. Then, for $g_\infty \in G(\mathbb{R})$,*

$$F_U(\varphi, \psi; (g_\infty; \mathbb{1})) = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(u(g_\infty; \mathbb{1}))\psi^{-1}(u)\, du = \int\limits_{U(\mathbb{Z})\backslash U(\mathbb{R})} f(u_\infty g_\infty)\psi_\infty^{-1}(u_\infty)\, du_\infty \tag{3.51}$$

which is a Fourier coefficient of f with respect to $U(\mathbb{R})$.

We now consider a Whittaker coefficient W_N and character ψ with rational charges $m_\alpha \in \mathbb{Q}$. Let $\hat{n} = (\mathbb{1}; \hat{n}_2, \hat{n}_3, \ldots) \in N(\hat{\mathbb{Z}}) \subset K_\mathbb{A}$. Then, for a spherical automorphic form φ and $a \in A(\mathbb{A})$ we have that $a\hat{n}a^{-1} \in N(\mathbb{A})$, and thus

$$W_N(\varphi, \psi; a) = W_N(\varphi, \psi; a\hat{n}) = W_N(\varphi, \psi; a\hat{n}a^{-1}a) = \psi(a\hat{n}a^{-1}) W_N(\varphi, \psi; a) \quad (3.52)$$

which implies that $\psi(a\hat{n}a^{-1}) = 1$ for $W_N(\varphi, \psi; a)$ to be non-vanishing. Restricting to $a = (a_\infty; \mathbb{1})$ with $a_\infty \in A(\mathbb{R})$ we find that

$$\psi(a\hat{n}a^{-1}) = \psi((a_\infty; \mathbb{1})(\mathbb{1}; \hat{n}_p)(a_\infty^{-1}; \mathbb{1})) = \psi(\hat{n}) = \exp\Big(-2\pi i \sum_{\substack{\alpha\in\Delta(\mathfrak{u}^{\mathrm{ab}})\\ p<\infty}} [m_\alpha \hat{n}_{\alpha,p}]_p\Big) \quad (3.53)$$

where we have parametrised $\hat{n}$ as $\hat{n} = \prod_{\alpha\in\Delta(\mathfrak{u}^{\mathrm{ab}})} x_\alpha(\hat{n}_\alpha)$ with $\hat{n}_\alpha = (1; \hat{n}_{\alpha,p}) \in \hat{\mathbb{Z}}$.

That $\psi(a\hat{n}a^{-1}) = 1$ for all $\hat{n}_{\alpha,p} \in \mathbb{Z}_p$ then implies that $\sum_{p<\infty}[m_\alpha]_p \in \mathbb{Z}$ for all $\alpha \in \Delta(\mathfrak{u}^{\mathrm{ab}})$. According to (3.5), this means that $m_\alpha \in \mathbb{Z}$, for W_N to be non-vanishing on $(a_\infty; \mathbb{1})$, which is expected since the restriction recovers the Whittaker coefficient for the real group $N(\mathbb{R})$.

Let P be a parabolic subgroup of G with Levi decomposition $P = LU$. The unipotent radical U is stabilised by the Levi subgroup L under conjugation. Let ψ be a character on $U(\mathbb{Q})\backslash U(\mathbb{A})$, $l \in L(\mathbb{Q})$ and define the character ψ^l on $U(\mathbb{Q})\backslash U(\mathbb{A})$ as

$$\psi^l(u) = \psi(lul^{-1})\,. \quad (3.54)$$

We will call ψ^l a conjugated or twisted character.

For an automorphic form $\varphi \in \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ and an element $l \in L(\mathbb{Q})$, we have from (III-1.5) that

$$F_U(\varphi, \psi^l; g) = F_U(\varphi, \psi; lg) \quad (3.55)$$

which means that we only need to compute the Fourier coefficient of one character ψ in the $L(\mathbb{Q})$-orbit $\{\psi^l \mid l \in L(\mathbb{Q})\}$, called a character variety orbit.

We will discuss these orbits further in the next section using the language of Whittaker pairs.

3.7 Nilpotent orbits and the wave-front set

When relating Fourier coefficients with respect to different unipotent subgroups U (as, for example, in Paper III) it is convenient to introduce the notation of a Whittaker pair, which specifies a unipotent subgroup and a character ψ as follows based on [42].

Let G be a reductive group over $\mathbb{Q}$ and $\mathfrak{g}$ the Lie algebra of $G(\mathbb{Q})$. For a semisimple element $s \in \mathfrak{g}$, let $\mathfrak{g}^s_i = \{x \in \mathfrak{g} \mid [s, x] = ix\}$, $\mathfrak{g}^s_{\geq i} = \oplus_{i'\geq i}\mathfrak{g}^s_{i'}$ and similarly for other relations. Denote also the centraliser of an element $X \in \mathfrak{g}$ in $\mathfrak{g}$ as $\mathfrak{g}_X$. We say that a semisimple element s is rational semisimple if all eigenvalues i are rational.

A Whittaker pair is then an ordered pair (s, u) of a rational semisimple element s and a nilpotent element $u \in \mathfrak{g}^s_{-2}$. If $s \in \mathrm{Im}(\mathrm{ad}(u))$, then (s, u) is called a neutral pair and is part of an $\mathfrak{sl}_2$-triple (u, s, v) satisfying the standard $\mathfrak{sl}_2$ commutation relations where $v \in \mathfrak{g}_2^s$. The Jacobson–Morozov theorem states that the $G(\mathbb{Q})$-conjugacy classes of $\mathfrak{sl}_2$ triples are in bijection with nilpotent orbits in $\mathfrak{g}$: $\mathcal{O}_X = \{gXg^{-1} \mid g \in G(\mathbb{Q})\}$ [108].

Let $\mathfrak{n}_s$ be the nilpotent subalgebra $\mathfrak{n}_s = \mathfrak{g}^s_{>1} \oplus \mathfrak{g}^s_1 \cap \mathfrak{g}_u$ of $\mathfrak{g}$ and N_s the corresponding unipotent subgroup of G. Define also the multiplicative character $\psi_u : N_s(\mathbb{Q})\backslash N_s(\mathbb{A}) \to U(1)$ as

$$\psi_u(n) = \mathbf{e}(\langle u \,|\, \log(n)\rangle) \qquad n \in N_s(\mathbb{A}) \tag{3.56}$$

with the Killing form $\langle\cdot|\cdot\rangle$. Note that $u \in \mathfrak{g}^s_{-2}$ is not an element of the unipotent group N_s.

We then define a Fourier coefficient associated with the Whittaker pair (s, u) as

$$F_{s,u}(\varphi; g) = F_{N_s}(\varphi, \psi_u; g) = \int\limits_{N_s(\mathbb{Q})\backslash N_s(\mathbb{A})} \varphi(ng)\psi_u^{-1}(ng)\, dn \tag{3.57}$$

for an automorphic form φ.

When (s, u) is a neutral pair, we have from (III-2.1) that $\mathfrak{n}_s = \mathfrak{g}^s_{\geq 2}$ and, for an element $\gamma \in G(\mathbb{Q})$, the Fourier coefficient associated with the neutral pair $(s', u') = (\gamma s \gamma^{-1}, \gamma u \gamma^{-1})$ is, according to (III-2.2),

$$F_{s',u'}(\varphi; g) = F_{s,u}(\varphi; \gamma^{-1} g)\,. \tag{3.58}$$

We will now use the vanishing properties of such Fourier coefficients to define minimal and next-to-minimal automorphic representations. Let π be an automorphic representation and let $F_{s,u}(\pi) = \{F_{s,u}(\varphi) \mid \varphi \in \pi\}$. The global wave-front set $\mathcal{WF}(\pi)$ is then defined to be the set of nilpotent orbits $\mathcal{O}$ such that $F_{s,u}(\pi)$ is non-vanishing for any neutral Whittaker pair (s, u) where $u \in \mathcal{O}$.

We can give nilpotent orbits a partial ordering by $\mathcal{O}' \leq \mathcal{O}$ if $\overline{\mathcal{O}'} \subseteq \overline{\mathcal{O}}$ where $\overline{\mathcal{O}}$ is the Zariski closure of $\mathcal{O}$. The Zariski topology is induced from that of the algebraically closed field $\mathbb{C}$, which is why we will often only need to specify a complex orbit when discussing the partial ordering. As seen in section III-2, there may be several rational orbits in each complex orbit.

There is a unique (non-trivial, complex) minimal orbit $\mathcal{O}_{\text{min}}$ and a unique next-to-minimal orbit $\mathcal{O}_{\text{ntm}}$ with respect to this partial ordering. An automorphic representation π is then said to be a minimal automorphic representation if $\mathcal{WF}(\pi)$ is the set of orbits in the closure of $\mathcal{O}_{\text{min}}$ and similarly for a next-to-minimal automorphic representation and $\mathcal{O}_{\text{ntm}}$.

Colloquially, we have that small automorphic representations have very few non-vanishing Fourier coefficients. This is particularly interesting for the four-graviton amplitude with the R^4 and D^4R^4 coefficients $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ in chapter 2 which are attached to a minimal and next-to-minimal automorphic representation respectively.

From [42, Theorem C] we have that if $F_{(h,u)}(\pi)$ is zero for a neutral pair (h, u), then $F_{(s,u)}(\pi)$ is zero for any Whittaker pair (s, u). This allows us to immediately determine if a Fourier coefficient of an automorphic form φ is vanishing based on the wave-front set associated to φ.

3.7.1 BPS-orbits

We will now connect back to the discussion in section 2.4.4 about extracting physical information from Fourier coefficients of automorphic forms. There we saw that, in the decompactification limit from D to $D+1$ dimensions, the Fourier characters in D dimensions are parametrised by the charges of BPS-particles in $D+1$ dimensions and the BPS-orbits correspond to the character variety orbits.

For example when $D=4$, the R^4 and D^4R^4 coefficients $\mathcal{E}^{(4)}_{(0,0)}$ and $\mathcal{E}^{(4)}_{(1,0)}$ carry information about $D=5$ BPS-states which can be studied by computing Fourier coefficients in the decompactification limit corresponding to a maximal parabolic subgroup whose Levi subgroup is $L = GL_1 \times E_6$. This Levi subgroup acts on the nilpotent subalgebra $\mathfrak{u} = \mathrm{Lie}(U)$ by conjugation furnishing a 27-dimensional vector representation of E_6, and similarly for the charges m_α parametrising the characters on $U(\mathbb{Q})\backslash U(\mathbb{A})$.

The R^4 coefficient $\mathcal{E}^{(4)}_{(0,0)}$ is attached to a minimal automorphic representation with a wave-front set being the set of orbits contained in the closure of the minimal orbit, that is, the trivial orbit and the minimal orbit, the latter of which intersects the $\frac{1}{2}$-BPS orbit. This means that the only non-trivial Fourier coefficients that contribute to the automorphic form in the decompactification limit are those with charges in the $\frac{1}{2}$-BPS orbit, which is why the R^4 coefficient is said to only receive contributions from $\frac{1}{2}$-BPS states.

Similarly the D^4R^4 coefficient $\mathcal{E}^{(4)}_{(1,0)}$ is attached to a next-to-minimal automorphic representation. The wave-front set is: the trivial orbit, the minimal orbit and the next-to-minimal orbit (which corresponds to the $\frac{1}{4}$-BPS orbit) and thus, $\mathcal{E}^{(4)}_{(1,0)}$ gets contributions from $\frac{1}{4}$-BPS states as well as $\frac{1}{2}$-BPS states.

Although the D^6R^4 coefficient $\mathcal{E}^{(4)}_{(0,1)}$ is not strictly an automorphic form and therefore cannot be associated to an automorphic representation, we may still consider the vanishing properties of its Fourier coefficients. Generally, the D^6R^4 coefficient gets contributions from $\frac{1}{8}$-BPS states as well as $\frac{1}{4}$-BPS and $\frac{1}{2}$-BPS states [18,40]. Note that, for $D=5$, this would include all possible charges, which means that the D^6R^4 coefficient shares the same vanishing properties as non-BPS protected coefficients at higher orders in α'. It is, however, expected that the Fourier coefficients take a simpler form compared to those for the higher-order coefficients $\mathcal{E}^{(D)}_{(p,q)}$. For more information about the D^6R^4 coefficient see section I-14.1.

Again, a similar story for the automorphic representations and vanishing properties of the R^4, D^4R^4 and D^6R^4 coefficients follows for other dimensions.

Chapter 4

Main results

In Paper I we review the existing theory of automorphic forms and representations as well as the related topic of classical modular forms together with the necessary mathematical background for both. We discuss connections with both string theory (an important example being the scattering amplitudes discussed in section 2.4 in this thesis), statistical mechanics as seen for example in [127–130], the Langlands' program, and extensions to Kac-Moody groups among other topics.

For this thesis, we will, in particular, highlight the results for computing adelic Whittaker coefficients of spherical Eisenstein series in section 4.1. These Fourier coefficients with respect to the unipotent radical N of the Borel subgroup will be the foundation for computing more complicated Fourier coefficients on other parabolic subgroups such as those of interest in string theory shown in table 2.3.

Many of the theorems in Paper I are well known in the literature (which we cite accordingly in the theorem name), but to our knowledge, Paper I is the first time the statements and proofs have been made available together in a cohesive program for computing Whittaker coefficients of spherical Eisenstein series. Although most probably known by the experts in the field, we also believe that the reduction method in theorem I-9.4 for computing degenerate Whittaker coefficients made its first public appearance in [131] by my collaborators of Paper I.

First, the constant terms, for which the character ψ is trivial, can be computed using Langlands' constant term formula proven in chapter I-8. Then, an unramified Whittaker coefficient where $\psi(\exp(uE_\alpha)) = \mathbf{e}(u)$ for $u \in \mathbb{A}$ and all $\alpha \in \Pi$ can be computed using the Casselman–Shalika formula from which we then can obtain generic Whittaker coefficients. Finally, by a reduction formula to smaller groups, the degenerate Whittaker coefficients can be computed where we note that the more degenerate the character is, the smaller the reduced group becomes, giving a less complicated end result.

In Paper II we derive methods for computing Fourier coefficients with respect of maximal parabolic subgroups of automorphic forms attached to small automorphic representations of SL_3 and SL_4 showed in section 4.2 following a construction by Ginzburg [45] to create standard Fourier coefficients associated to the different nilpotent orbits of the group and which are known to vanish if the orbit is not in the global wave-front set. To understand the vanishing properties of the automorphic forms and the wanted maximal parabolic Fourier coefficients we express them in terms of these orbit Fourier coefficients. After that, the orbit Fourier coefficients are determined in terms of Whittaker coefficients which can be computed using the methods of Paper I. In particular we find that maximal parabolic Fourier coefficients of automorphic forms attached to a minimal representation is equal to an $L(\mathbb{Q})$-translated maximally degenerate Whittaker coefficient.

We also obtain expressions for the complete expansion of an automorphic form for these groups in Paper II in terms of Whittaker coefficients as described in section 4.3. We show that an automorphic form attached to a minimal automorphic representation is determined by Whittaker coefficients with support on at most a single simple root and similarly for a next-to-minimal representation and Whittaker coefficients with support on at most two commuting simple roots.

In Paper III we consider SL_n for $n \geq 5$ and use the notion of Whittaker pairs and theorem III-2.1 [42, Theorem C] to immediately determine the vanishing properties of a Fourier coefficient and, for expressing these in terms of Whittaker coefficients, we now also use the root exchange lemma III-2.5 by [52]. We again find that the maximal parabolic Fourier coefficients are equal to maximally degenerate Whittaker coefficients for a minimal automorphic representation, but we also obtain expressions for the case of a next-to-minimal representation in terms of Whittaker coefficients supported on at most two commuting simple roots. Similar to SL_3 and SL_4 we also write the complete automorphic form in terms of Whittaker coefficients for small automorphic representations.

Finally, we also have some results for computing maximal parabolic Fourier coefficients for automorphic forms attached to a minimal automorphic representation of E_6, E_7 or E_8 from Paper II as discussed in 4.2.

4.1 Whittaker coefficients

It is important to note that we may consider the Fourier expansion of an Eisenstein series with respect to any parabolic subgroups even though the Eisenstein series itself is defined by a character χ on another parabolic subgroup. As seen in section 3.4 we can relate an Eisenstein series defined by a character on a (non-minimal) parabolic subgroup with an Eisenstein series on the Borel subgroup which is why we here only need to consider characters χ on the Borel subgroup.

Let ψ be a character on the unipotent radical N of B and consider the Whittaker coefficient of a spherical Eisenstein series $E(\chi; g)$ where χ is a character on B

$$W(\chi, \psi; g) = \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi; ng)\psi^{-1}(n)\, dn \tag{4.1}$$

where we have dropped the subscript N

Following section I-9.5, the Bruhat decomposition [132]

$$G(\mathbb{Q}) = \bigcup_{w \in \mathcal{W}} B(\mathbb{Q})wB(\mathbb{Q}) \tag{4.2}$$

where $\mathcal{W}$ is the Weyl group of $G(\mathbb{R})$ and B is the Borel subgroup of G, can be used to rewrite the sum over cosets in the Eisenstein series.

Define

$$\begin{aligned} \mathcal{C}_\psi &= \{w \in \mathcal{W} \mid w\alpha < 0 \text{ for all } \alpha \in \operatorname{supp}(\psi)\} \\ N^{(w)}(\mathbb{A}) &= \prod_{\substack{\alpha\in\Delta_+\\ w\alpha<0}} X_\alpha(\mathbb{A}) \end{aligned} \tag{4.3}$$

where $\operatorname{supp}(\psi) = \{\alpha \in \Pi \mid \psi|_{X_\alpha} \not\equiv 1\}$ and $X_\alpha(\mathbb{A}) = \{\exp(xE_\alpha) \mid x \in \mathbb{A}\}$.

The Bruhat decomposition splits the Whittaker coefficient into a sum over Weyl words where each term factorises with one of the factors vanishing for Weyl words not in $\mathcal{C}_\psi$. The remaining terms become

$$\begin{aligned} W(\chi,\psi;a) &= \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi,na)\psi^{-1}(n)\,dn = \sum_{w\in\mathcal{C}_\psi} F_w(\chi,\psi;a) \\ F_w(\chi,\psi;a) &= \int\limits_{N^{(w)}(\mathbb{A})} \chi(wna)\psi^{-1}(n)\,dn\,. \end{aligned} \tag{4.4}$$

We are left with adelic integrals which factorise as

$$\begin{aligned} F_w(\chi,\psi;a) &= \prod_{p\le\infty} F_{w,p}(\chi_p,\psi_p;a_p) \\ F_{w,p}(\chi_p,\psi_p;a_p) &= \int\limits_{N^{(w)}(\mathbb{Q}_p)} \chi_p(wna_p)\psi_p^{-1}(n)\,dn\,. \end{aligned} \tag{4.5}$$

For an element $\tilde{a} \in A(\mathbb{A})$ we will call $\psi^{\tilde{a}}$, defined by $\psi^{\tilde{a}}(n) = \psi(\tilde{a}n\tilde{a}^{-1})$ for $n \in N(\mathbb{A})$, a twisted character, although it may no longer be trivial on $N(\mathbb{Q})$. In this case, we will by $W_N(\chi,\psi^{\tilde{a}};a)$ mean the sum over the integrals $F_w(\chi,\psi^{\tilde{a}};a)$ which are well-defined having integration domains $N^{(w)}(\mathbb{A})$ instead of $N(\mathbb{Q})\backslash N(\mathbb{A})$.

Let us start by considering the case when ψ is trivial, that is, $\psi = 1$. For a trivial character $\mathcal{C}_1 = \mathcal{W}$ since $\operatorname{supp}(1) = \emptyset$.

Theorem I-8.1. (Langlands' constant term formula [124]). *Let χ be a character on the Borel subgroup $B(\mathbb{A}) \subset G(\mathbb{A})$ trivial on the unipotent radical N and parametrised by a weight $\lambda \in \mathfrak{h}^*_\mathbb{C}$ as $\chi(g) = \chi(nak) = |a|^{\lambda+\rho}$. Then, the constant term of the Eisenstein series $E(\chi;g)$ with respect to N is*

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi;ng)\,dn = \sum_{w\in\mathcal{W}} |a|^{w\lambda+\rho} M(w,\lambda) \qquad M(w,\lambda) = \prod_{\substack{\alpha\in\Delta_+\\ w\alpha<0}} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(1+\langle\lambda|\alpha\rangle)}\,. \tag{4.6}$$

The proof of theorem I-8.1 is shown in chapter I-8. For the constant terms with respect to the unipotent radical of maximal parabolic subgroups see theorem I-8.9 presented in section 4.2.

For non-trivial characters we will first consider generic characters, that is, characters ψ such that $\operatorname{supp}(\psi) = \Pi$. In that case we have that $\mathcal{C}_\psi = \{w_{\text{long}}\}$

since only the longest Weyl word w_{long} reflects Π to $-\Pi$, and we only need to compute $F_{w_{\text{long}}}$ from (4.3) with $N^{(w_{\text{long}})} = N$. When ψ is unramified, that is $\psi = \hat{\psi}$ which is defined by $\hat{\psi}(\exp(uE_\alpha)) = \mathbf{e}(u)$ for $u \in \mathbb{A}$ and all $\alpha \in \Pi$, the p-adic local factors $F_{w_{\text{long}},p}$ with $p < \infty$ can be computed using the Casselman–Shalika formula.

Theorem I-9.1. (The Casselman–Shalika formula [133]). *Let χ be a character on the Borel subgroup $B(\mathbb{A}) \subset G(\mathbb{A})$ trivial on the unipotent nilradical N and parametrised by a weight $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ as $\chi(g) = \chi(nak) = |a|^{\lambda+\rho}$ factorising as $\chi(a) = \prod_{p\leq\infty} \chi_p(a_p) = \prod_{p\leq\infty} |a_p|_p^{\lambda+\rho}$. Then, for $p < \infty$,*

$$F_{w_{\text{long}},p}(\chi_p, \hat{\psi}_p; a_p) = \int\limits_{N(\mathbb{Q}_p)} \chi_p(w_{\text{long}} n a_p) \hat{\psi}_p^{-1}(n)\, dn = \frac{1}{\zeta(\lambda)} \sum_{w\in\mathcal{W}} \epsilon(w\lambda)\, |a|_p^{w\lambda+\rho} \tag{4.7}$$

where

$$\zeta(\lambda) = \prod_{\alpha\in\Delta_+} \frac{1}{1 - p^{-(\langle\lambda|\alpha\rangle+1)}} \qquad \epsilon(\lambda) = \prod_{\alpha\in\Delta_+} \frac{1}{1 - p^{\langle\lambda|\alpha\rangle}}\,. \tag{4.8}$$

The remaining factor $F_{w_{\text{long}},\infty}$ needs to be computed by hand. The unramified Whittaker coefficient is then

$$W(\chi, \hat{\psi}; a) = \prod_{p\leq\infty} F_{w_{\text{long}},p}(\chi_p, \hat{\psi}_p; a_p)\,. \tag{4.9}$$

We will now consider a generic character ψ satisfying $\operatorname{supp}(\psi) = \Pi$. Let $n \in N(\mathbb{A})$ be parametrised as

$$n = \Big(\prod_{\alpha\in\Delta\setminus\Pi} x_\alpha(u_\alpha)\Big)\Big(\prod_{\alpha\in\Pi} x_\alpha(u_\alpha)\Big)\,. \tag{4.10}$$

As in section 3.6, we can then parametrise the character ψ on $N(\mathbb{A})$ by charges $m_\alpha \in \mathbb{Q}$ as

$$\psi(n) = \mathbf{e}\Big(\sum_{\alpha\in\Pi} m_\alpha u_\alpha\Big)\,. \tag{4.11}$$

Let A_{ij} be the Cartan matrix of the group G with rank r, and $\tilde{a} \in A(\mathbb{A})$ parametrised by

$$\tilde{a} = \prod_{i=1}^{r} h_{\alpha_i}(t_{\alpha_i}) \qquad t_{\alpha_i} \in \mathbb{A}^\times\,. \tag{4.12}$$

Then, $\tilde{a}n\tilde{a}^{-1}$ becomes factors of $\tilde{a}x_\beta(u)\tilde{a}^{-1}$ by insertions of $\tilde{a}\tilde{a}^{-1}$ in (4.10). From [113] we have that, for a simple root α, and a root β,

$$h_\alpha(t) x_\beta(u) h_\alpha(t)^{-1} = x_\beta(t^{\beta(H_\alpha)} u)\,. \tag{4.13}$$

When determining $\psi(\tilde{a}n\tilde{a}^{-1})$, we then only need to consider the factors with $\beta \in \Pi$ for which

$$h_{\alpha_i}(t) x_{\alpha_j}(u) h_{\alpha_i}(t)^{-1} = x_{\alpha_j}(t^{A_{ij}} u)\,. \tag{4.14}$$

Thus,

$$\tilde{a}x_{\alpha_j}(u)\tilde{a}^{-1} = x_{\alpha_j}\Big(\prod_{i=1}^{r}(t_{\alpha_i})^{A_{ij}}u\Big) \tag{4.15}$$

and

$$\hat{\psi}^{\tilde{a}}(n) = \hat{\psi}(\tilde{a}n\tilde{a}^{-1}) = \mathbf{e}\Big(\sum_{j=1}^{r}\prod_{i=1}^{r}(t_{\alpha_i})^{A_{ij}}u_{\alpha_j}\Big) \tag{4.16}$$

which is a generic character on $U(\mathbb{Q})\backslash U(\mathbb{A})$ if $\prod_{i=1}^{r}(t_{\alpha_i})^{A_{ij}} \in \mathbb{Q}$ for all j. We would then, with a variable substitution similar to that in (I-9.18), obtain that

$$\begin{aligned} W_N(\chi, \hat{\psi}^{\tilde{a}}; a) &= |\tilde{a}|^{-w_{\text{long}}\lambda-\rho}\, W_N(\chi, \hat{\psi}; \tilde{a}a) \\ &= \prod_{p\leq\infty}\prod_{\alpha\in\Pi} |t_\alpha|_p^{-\langle w_{\text{long}}\lambda+\rho|H_\alpha\rangle}\, F_{w_{\text{long}},p}(\chi_p, \hat{\psi}_p; \tilde{a}_p a_p)\,. \end{aligned} \tag{4.17}$$

The factors for finite p are, according to (4.7),

$$\begin{aligned} \prod_{\alpha\in\Pi} |t_\alpha|_p^{-\langle w_{\text{long}}\lambda+\rho|H_\alpha\rangle}\, \frac{1}{\zeta(\lambda)} \sum_{w\in\mathcal{W}} \epsilon(w\lambda)\, |\tilde{a}|_p^{w\lambda+\rho}\, |a|_p^{w\lambda+\rho} = \\ = \frac{1}{\zeta(\lambda)} \sum_{w\in\mathcal{W}} \epsilon(w\lambda) \prod_{\alpha\in\Pi} |t_\alpha|_p^{\langle(w-w_{\text{long}})\lambda|H_\alpha\rangle}\, |a|_p^{w\lambda+\rho} \end{aligned} \tag{4.18}$$

This gives us an expression for a generic Whittaker coefficient with character ψ parametrised by $m_\alpha \in \mathbb{Q}$ by (4.11) if $m_{\alpha_j} = \prod_{i=1}^{r}(t_{\alpha_i})^{A_{ij}}$ can be solved with $t_{\alpha_i} \in \mathbb{A}^\times$ for all j. Note that the equation is always solvable in $\mathbb{R}$ as

$$t_{\alpha_i,\infty} = \prod_{j=1}^{r}(m_{\alpha_j})^{(A^{-1})_{ji}} \tag{4.19}$$

and that the right-hand-side of (4.18) is well-defined if we formally make the replacement

$$|t_{\alpha_i}|_p \to \prod_{j=1}^{r}\big|m_{\alpha_j}\big|_p^{(A^{-1})_{ji}} \tag{4.20}$$

although the intermediate steps, as presented here, assume that $\tilde{a} \in A(\mathbb{A})$.

Lastly, we turn to degenerate Whittaker coefficients with characters ψ such that $\operatorname{supp}(\psi) = \Pi' \subsetneq \Pi$ leading to a larger $\mathcal{C}_\psi$ than for the generic case. Let $G'(\mathbb{A}) \subset G(\mathbb{A})$ be the group associated to the simple roots Π' and $\mathcal{W}' \subset \mathcal{W}$ the corresponding Weyl group with longest Weyl word w'_{long}. This Weyl word w'_{long} is the unique word in $\mathcal{W}'$ which maps all simple roots in Π' to negative roots, and any element $w \in \mathcal{C}_\psi$ can be represented as $w = w_c w'_{\text{long}}$ for some $w_c \in \mathcal{W}$ such that the roots $w_c\Pi'$ are positive. In section I-9.5 it is shown how to obtain the elements w_c as carefully chosen representatives of $\mathcal{W}/\mathcal{W}'$.

For such $w_c w'_{\text{long}}$ we then have that

$$W(\chi, \psi; a) = \sum_{w_c w'_{\text{long}}\in\mathcal{W}/\mathcal{W}'} F_{w_c w'_{\text{long}}}(\chi, \psi; a)\,. \tag{4.21}$$

As is also shown in section I-9.5, the integration domain $N^{(w_c w'_{\text{long}})}(\mathbb{A})$ in each term $F_{w_c w'_{\text{long}}}$ can be factorised into two parts. From the first part we obtain a generic Whittaker coefficient on the subgroup $G'(\mathbb{A})$ where we recall that $\operatorname{supp}(\psi) = \Pi'$. The second part contains additional factors similar to those in the Langlands constant term formula.

Theorem I-9.4. *Let χ be a character on the Borel subgroup $B(\mathbb{A}) \subset G(\mathbb{A})$ trivial on the unipotent nilradical N and parametrised by a weight $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ as $\chi(g) = \chi(nak) = |a|^{\lambda+\rho}$. Let also ψ be a degenerate character with* $\operatorname{supp}(\psi) = \Pi' \subsetneq \Pi$ *with group $G'(\mathbb{A})$, Weyl group $\mathcal{W}'$ and representatives $w_c w'_{\text{long}}$ of $\mathcal{C}_\psi$ as described above. Then,*

$$W(\chi, \psi; a) = \sum_{w_c w'_{\text{long}} \in \mathcal{W}/\mathcal{W}'} |a|^{(w_c w'_{\text{long}})^{-1}\lambda+\rho} M(w_c^{-1}, \lambda) W'(\lambda', \psi^a; \mathbb{1}) \tag{4.22}$$

where $M(w, \lambda)$ is defined in (4.6)*, W' is a generic Whittaker coefficient on $G'(\mathbb{A})$ and the weight λ' is the orthogonal projection of $w_c^{-1}\lambda$ on the weight space of $G'(\mathbb{A})$.*

4.2 Fourier coefficients on parabolic subgroups

Let us now consider Fourier coefficients with respect to the unipotent radicals of parabolic subgroups other than the Borel subgroup. In particular, we will focus on maximal parabolic subgroups which are of great importance in string theory as discussed in section 2.4.4.

Fix a maximal parabolic subgroup P_{α_j} with $\Sigma = \Pi \setminus \{\alpha_j\}$ as described in section 3.2 with unipotent radical U and Levi subgroup L. The subrootsystem $\langle\Sigma\rangle$ describes a subgroup G' of G with Weyl group $\mathcal{W}'$ and we may write the Levi subgroup as $L = GL_1 \times G'$. Define also the projections

$$\begin{aligned} \Pi_j : \mathfrak{h}^*_{\mathbb{C}} &\to \mathfrak{h}^*_{\mathbb{C}} \\ \lambda &\mapsto \frac{\langle \Lambda_j | \lambda \rangle}{\langle \Lambda_j | \Lambda_j \rangle} \Lambda_j \,, \\ \Pi_j^\perp &= 1 - \Pi_j \end{aligned} \tag{4.23}$$

where Λ_j is the fundamental weight for α_j in G.

Theorem I-8.9. (Constant term in maximal parabolics [134]). *Let χ be a character on the Borel subgroup $B(\mathbb{A}) \subset G(\mathbb{A})$ trivial on the unipotent nilradical N and parametrised by a weight $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ as $\chi(g) = \chi(nak) = |a|^{\lambda+\rho}$. Let also P_{α_j} be defined as above with unipotent radical U, and let $g \in G$ have the Iwasawa factorisation $g = nak$. Then,*

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\chi, ug)\, du = \sum_{w \in \mathcal{W}'\backslash\mathcal{W}} |a|^{\Pi_j(w\lambda+\rho)} M(w, \lambda) E'(\chi', g) \tag{4.24}$$

where $M(w, \lambda)$ is defined in (4.6) *and $E'(\chi', g)$ is an Eisenstein series on G' with character $\chi'(a) = |a|^{\Pi_j^\perp(w\lambda+\rho)}$ on the Borel subgroup of G'. Note that we use the argument g in E' for simplicity although it effectively only depends on G'.*

Remark 4.1. Theorem I-8.9 as stated in Paper I also holds for Eisenstein series defined with characters on arbitrary parabolic subgroups.

Let us now turn to the results from Paper II and Paper III where we focus on automorphic forms attached to small automorphic representations, mainly for the group SL_n but also with some results for E_6, E_7 and E_8. Paper III treats a general number field F, but we will here restrict to $F = \mathbb{Q}$ as for the other papers of this thesis. Since the constant mode can be computed using theorem I-8.9 presented above, we will only focus on non-trivial characters from now on. Note also that the results of Paper III holds for any automorphic form on $SL_n(\mathbb{A})$ and not only for spherical Eisenstein series.

Starting with SL_n where $n \geq 5$, let P_{α_m} be the maximal parabolic subgroup defined above and let ψ be a character on its unipotent radical U. For SL_n, the unipotent radical U can be visualised in terms of $n \times n$ matrices as an $m \times (n-m)$ block in the upper right corner.

The characters are parametrised by elements $y \in \overline{\mathfrak{u}}(\mathbb{Q}) = {}^t\mathfrak{u}(\mathbb{Q})$ by

$$\psi_y(u) = \mathbf{e}(\operatorname{tr}(y \log(u))) \qquad u \in U(\mathbb{A}) \tag{4.25}$$

where y, in turn, is parametrised by an $(n-m) \times m$ matrix in the lower left corner which we will call Y

$$y(Y) = \begin{pmatrix} I_m & 0 \\ Y & I_{m-n} \end{pmatrix} . \tag{4.26}$$

As seen in (3.55) we only need to compute one of the Fourier coefficients of each character variety orbit. Based on [135], the $L(\mathbb{Q})$-orbits of ${}^t\mathfrak{u}(\mathbb{Q})$ are described in Paper III with standard representatives $y(Y_r)$ where r denotes the rank of Y_r which is non-zero everywhere except for an $r \times r$ anti-diagonal matrix in its upper right corner with unit elements. $y(Y_1)$ and $y(Y_2)$ belong the minimal and next-to-minimal G-orbits respectively.

For the next theorem we will also need the following definitions. Let ψ be a character on the unipotent radical N of the Borel subgroup, T the diagonal elements of $SL_n(\mathbb{Q})$, and $T_\psi = \{h \in T \mid \psi(hnh^{-1}) = \psi(n)\, \forall n \in N(\mathbb{A})\}$.

Define

$$\begin{aligned} \Gamma_i(\psi) &= \begin{cases} (SL_{n-i})_{\hat{Y}}(\mathbb{Q})\backslash SL_{n-i}(\mathbb{Q}) & 1 \leq i \leq n-2 \\ (T_\psi \cap T_{\psi_{\alpha_{n-1}}})\backslash T_\psi & i = n-1 \end{cases} \\ \Lambda_j(\psi) &= \begin{cases} (SL_j)_{\hat{X}}(\mathbb{Q})\backslash SL_j(\mathbb{Q}) & 2 \leq j \leq n \\ (T_\psi \cap T_{\psi_{\alpha_1}})\backslash T_\psi & j = 2 \end{cases} \end{aligned} \tag{4.27}$$

where $(SL_{n-i})_{\hat{Y}}$ is the stabiliser of $\hat{Y} = {}^t(1, 0, \ldots, 0) \in \mathrm{Mat}_{(n-i)\times 1}(\mathbb{Q})$ and similarly for $(SL_j)_{\hat{X}}$ which is the stabiliser of $\hat{X} = (0, \ldots 0, 1) \in \mathrm{Mat}_{1\times j}(\mathbb{Q})$ with respect to multiplication on the right. For a simple root α, let ψ_α be the character on N such that its only non-zero charge is $m_\alpha = 1$, and let $\psi_{\alpha_{i_1},\ldots,\alpha_{i_m}} = \psi_{\alpha_{i_1}} \cdots \psi_{\alpha_{i_m}}$. Finally, we define the embeddings $\iota, \hat{\iota} : SL_{n-i} \to SL_n$ as

$$\iota(\gamma) = \begin{pmatrix} I_i & 0 \\ 0 & \gamma \end{pmatrix} \qquad \hat{\iota}(\gamma) = \begin{pmatrix} \gamma & 0 \\ 0 & I_i \end{pmatrix} . \tag{4.28}$$

Theorem III-B. *For $n \geq 5$, let π be a minimal or next-to-minimal irreducible automorphic representation of $SL_n(\mathbb{A})$ with r_π being 1 or 2 respectively, and let $\varphi \in \pi$. Let P_{α_m} be the maximal parabolic subgroup of SL_n with respect to the root α_m, L its Levi subgroup, U the unipotent radical of P_{α_m} and $\psi_{y(Y_r)}$ one of the standard characters on U as described above. Then, these standard characters cover all character variety orbits and are enough to compute any Fourier coefficient with respect to U. The Fourier coefficient $F_U(\varphi, \psi_{y(Y_r)}; g) = \int_{[U]} \varphi(ug)\psi^{-1}_{y(Y_r)}(u)\, du$ vanishes for $r > r_\pi$ and the remaining (non-constant) coefficients can be expressed in terms of Whittaker coefficients as follows.*

(i) *If $\pi = \pi_{min}$:*

$$\mathcal{F}_U(\varphi, \psi_{y(Y_1)}; g) = \int\limits_{[N]} \varphi(ng)\psi^{-1}_{\alpha_m}(n)\, dn \tag{4.29}$$

(ii) *If $\pi = \pi_{ntm}$:*

$$\begin{aligned} \mathcal{F}_U(\varphi, \psi_{y(Y_1)}; g) = & \int\limits_{[N]} \varphi(ug)\psi^{-1}_{\alpha_m}(u)\, du + \\ & + \sum_{j=1}^{m-2} \sum_{\gamma \in \Lambda_j(\psi_{\alpha_m})} \int\limits_{[N]} \varphi(u\hat{\iota}(\gamma)g)\psi^{-1}_{\alpha_m,\alpha_j}(u)\, du + \\ & + \sum_{i=m+2}^{n-1} \sum_{\gamma \in \Gamma_i(\psi_{\alpha_m})} \int\limits_{[N]} \varphi(u\iota(\gamma)g)\psi^{-1}_{\alpha_m,\alpha_i}(u)\, du\,. \end{aligned} \tag{4.30}$$

(iii) *If $\pi = \pi_{ntm}$:*

$$\mathcal{F}_U(\varphi, \psi_{y(Y_2)}; g) = \int\limits_{C(\mathbb{A})} \int\limits_{[N]} \varphi(n\omega cg)\psi^{-1}_{\alpha_1,\alpha_3}(n) dndc \tag{4.31}$$

where ω is the Weyl element that maps the torus elements $(t_1, t_2, \ldots, t_n)$ to $(t_{m-1}, t_{m+2}, t_m, t_{m+1}, t_1, t_2, \ldots, t_{m-2}, t_{m+3}, t_{m+4}, \ldots, t_n)$ and the subgroup C of U is defined in section III-4.

This means that, for a minimal representation, maximal parabolic Fourier coefficients can be expressed in terms of maximally degenerate Whittaker coefficients and, for a next-to-minimal representation, in terms of Whittaker coefficients with support on at most two commuting roots.

Since there is only one standard maximal parabolic for SL_2, which is also the Borel subgroup, this leaves only SL_3 and SL_4 which were treated in Paper II.

Theorem II-IV. *For $3 \leq n \leq 4$, let π be a minimal irreducible automorphic representation of $SL_n(\mathbb{A})$ with $\varphi \in \pi$ and let P_α be a maximal parabolic subgroup of SL_n with respect to the simple root α, L its Levi subgroup, U the unipotent radical and ψ a character on U. Then,*

$$\mathcal{F}_U(\varphi, \psi; g) = \int\limits_{[U]} \varphi(ug)\psi^{-1}(u)\, du = \int\limits_{[N]} \varphi(nlg)\psi_\alpha^{-1}(n)\, dn = W(\varphi, \psi_\alpha; lg) \tag{4.32}$$

for some $l \in L(\mathbb{Q})$ described in the proof in Paper II.

We have now shown that, for an automorphic form in a minimal automorphic representation of SL_n with $n \geq 2$, non-trivial maximal parabolic Fourier coefficients are $L(\mathbb{Q})$-translates of maximally degenerate Whittaker coefficients.

In Paper II, we also investigated whether this property would hold also for E_6, E_7 and E_8. Since the maximal parabolic Fourier coefficients for these groups are only known in a few cases, we instead explored the implications of such a property and could, in this way, make indirect checks.

The purpose of the ongoing project listed after the appended papers in the List of publications, is to compute the maximal Fourier coefficients of automorphic forms attached to small automorphic representations of these groups among others.

As seen in (4.9) and (4.18) generic Whittaker coefficients are Eulerian, that is, they factorise over $p \leq \infty$, but when $\operatorname{supp}(\psi) \neq \Pi$ for a character ψ on N, then $|\mathcal{C}_\psi| > 1$ and we get more than one term in (4.22), meaning that degenerate Whittaker coefficients do not factorise in general.

Thus, we cannot a priori assume that a general Fourier coefficient with respect to a maximal parabolic subgroup factorises, that is, we cannot expect that

$$\operatorname{Ind}_{U(\mathbb{A})}^{G(\mathbb{A})} \psi = \bigotimes_{p \leq \infty} \operatorname{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \psi_p\,, \tag{4.33}$$

where the notation is explained in section 3.5.

However, if, for a minimal automorphic representation of E_6, E_7 or E_8, Fourier coefficients with respect to maximal parabolic subgroups are $L(\mathbb{Q})$-translates of maximally degenerate Whittaker coefficients, which for E_6, E_7 and E_8 are listed in appendix A of [131] and can be seen to be Eulerian, then the maximal parabolic Fourier coefficient would indeed factorise.

We can then compare the local factors of these maximally degenerate Whittaker coefficients with elements in the image of the embedding

$$\pi_{\min,p} \subset \operatorname{Ind}_{P'(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_{\min,p} \hookrightarrow \operatorname{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \psi_p \tag{4.34}$$

where $\chi_{\min,p}$ is a certain spherical character on a standard parabolic subgroup $P'(\mathbb{Q}_p)$ (whose unipotent radical may be different from U).

The resulting space is of multiplicity one [136] which means that the elements of this space, called local spherical vectors, are unique up to normalisation and such local spherical vectors have been computed for several different parabolic subgroups of E_6, E_7, and E_8 in [14, 53–55] using methods from representation theory. The following proposition from Paper II compares these known local spherical vectors with the local factors of maximally degenerate Whittaker coefficients finding complete agreement. This provides strong evidence for that the Fourier coefficients of these maximal parabolic subgroups are indeed $L(\mathbb{Q})$-translates of maximally degenerate Whittaker coefficients.

Propositions II-1.1, 1.2 and 1.3. *Let $P' = P_{\alpha_1}$ and $\chi_{min}(a) = |a|^{\lambda+\rho}$ where $\lambda = 2s\Lambda_1 - \rho$. Let also*

$$P = P_{\alpha_j} = \begin{cases} P_{\alpha_1} \text{ or } P_{\alpha_2} & \text{for } E_6 \\ P_{\alpha_1} \text{ or } P_{\alpha_7} & \text{for } E_7 \\ P_{\alpha_8} & \text{for } E_8 \end{cases}, \tag{4.35}$$

and let U be the unipotent radical of P, and ψ a character on U.

Then, the local factors for both $p < \infty$ and $p = \infty$ of $W(\chi_{min}, \psi_{\alpha_i}; \mathbb{1})$ matches the unique (up to normalisation) local spherical vectors in $\mathrm{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \psi_p$ computed in [14, 53–55].

The Fourier expansion of $\mathcal{E}^{(D)}_{(0,0)}$ in the cases P_{α_7} and P_{α_8} for E_7 and E_8 respectively (among others) were later computed in [43] and takes the form of a sum of SL_2 Whittaker coefficients (seen in (2.79)) as expected if the maximal parabolic Fourier coefficients are translates of maximally degenerate Whittaker coefficients on E_7 or E_8 respectively.

In Paper III we also compute the real Fourier coefficients with respect to certain maximal parabolic subgroups for Eisenstein series in a minimal or next-to-minimal automorphic representation of SL_5 of interest in string theory by using theorem III-B, restricting the argument to $g = (g_\infty; \mathbb{1})$ and using the results of Paper I to evaluate the resulting Whittaker coefficients. These maximal parabolic Fourier coefficients carry information about non-perturbative effects in the decompactification limit of the seven-dimensionsal type IIB string theory as discussed in chapter 2. For the minimal case, our expression agrees with previous results from [18] with a translation of notation. Our result for the next-to-minimal case takes a different form compared to [18] as detailed in section III-6, and we are currently investigating non-trivial identities involving infinite sums of integrals of Bessel functions which may relate the two expressions.

Lastly, we also determine standard Fourier coefficients associated to the orbits $[21^{n-2}]$ and $[2^21^{n-4}]$ of SL_n in theorems III-C and III-D, in terms of maximal parabolic Fourier coefficients that were considered in theorem III-B.

4.3 Complete expansions

After developing the above methods for computing maximal parabolic Fourier coefficients in terms of Whittaker coefficients for automorphic forms attached to small automorphic representations we may also apply similar methods to determine the complete expansion of an automorphic form in terms of Whittaker coefficients.

Theorem III-A. *Let π be a minimal or next-to-minimal irreducible automorphic representation of $SL_n(\mathbb{A})$ for $n \geq 5$, and let $\varphi \in \pi$.*

(i) *If $\pi = \pi_{min}$, then φ has the expansion*

$$\varphi(g) = \int\limits_{[N]} \varphi(ng)\, dn + \sum_{i=1}^{n-1} \sum_{\gamma \in \Gamma_i} \int\limits_{[N]} \varphi(n\iota(\gamma)g)\psi_{\alpha_i}(n)\, dn\,. \tag{4.36}$$

(ii) *If $\pi = \pi_{ntm}$, then φ has the expansion*

$$\begin{aligned}\varphi(g) = \int\limits_{[N]} \varphi(vg)\, dv + \sum_{i=1}^{n-1} \sum_{\gamma \in \Gamma_i} \int\limits_{[N]} \varphi(v\iota(\gamma)g)\psi_{\alpha_i}^{-1}(v)\, dv \, + \\ + \sum_{j=1}^{n-3} \sum_{i=j+2}^{n-1} \sum_{\substack{\gamma_i \in \Gamma_i(\psi_{\alpha_j}) \\ \gamma_j \in \Gamma_j}} \int\limits_{[N]} \varphi(v\iota(\gamma_i)\iota(\gamma_j)g)\psi_{\alpha_j,\alpha_i}^{-1}(v)\, dv\end{aligned} \tag{4.37}$$

where $\Gamma_i, \Lambda_j, \iota$ and $\hat{\iota}$ are defined in section 4.2.

Theorems II-I, II and III. *Let π be an irreducible automorphic representation of $SL_3(\mathbb{A})$ or $SL_4(\mathbb{A})$ and $\varphi \in \pi$. Then,*

$$\varphi(g) = \sum_{\mathcal{O}} \mathcal{F}_{\mathcal{O}}(\varphi, g) \tag{4.38}$$

where the sum is over nilpotent orbits $\mathcal{O}$ and each term $\mathcal{F}_{\mathcal{O}}$ vanishes unless $\mathcal{O} \in \mathcal{WF}(\pi)$ and each $\mathcal{F}_{\mathcal{O}}(\varphi, g)$ is computed in terms of Whittaker coefficients in sections II-3.2 and 4.2. In particular, if π is a minimal or next-to-minimal automorphic representation, φ is completely determined by Whittaker coefficients supported on at most a single simple root or at most two commuting simple roots respectively.

Using the methods of Paper III, a theorem similar to the one directly above may also be proven for SL_n with $n \geq 5$.

4.4 Structure of proofs

In this section we will cover the structure of the proofs for the main theorems of Paper II and Paper III as well as the reduction formula of theorem I-9.4. The other theorems presented in this chapter are known from the literature, and while we have presented the overarching ideas behind the theorems in the previous sections where the theorems are stated, we leave further details to the found in Paper I and the respective references.

Theorems II-I, II and III were proved by first constructing Fourier coefficients attached to different nilpotent orbits via a construction by Ginzburg [45]. The vanishing properties of these particular orbit Fourier coefficients were then known from the wave-front set.

The automorphic form was Fourier expanded with respect to the unipotent radical of the Borel subgroups together with the commutator subgroups. The Whittaker coefficients (and their related coefficients on the commutator subgroups) were then related to the above orbit Fourier coefficients giving an expansion of the automorphic forms in terms of the latter, thus proving the theorems.

Theorem II-IV was proven in a similar way by relating maximal parabolic Fourier coefficients with the above orbit Fourier coefficients to determine their vanishing properties. The orbit Fourier coefficients were then expressed in terms of Whittaker coefficients.

In Paper III, theorem III-A takes a more systematic approach by expanding the automorphic form in row by row on the unipotent radical of the Borel (which, for SL_n can be seen as the upper triangular matrices). By using $L(\mathbb{Q})$-conjugations each row expansion could be put on a standard form with characters supported only on the simple roots.

Using [42, Theorem C] and the notion of Whittaker pairs (both described in section 3.7), the vanishing properties of the iteratively obtained Fourier coefficients could be immediately determined without the explicit construction of the orbit Fourier coefficients from [45].

This theorem may be compared with the Piatetski-Shapiro and Shalika formula [46, 47] which uses a similar expansion along rows (or columns) to find a Fourier expansion of cusp forms on GL_n.

In the proof of theorem III-B, we started with the unipotent radical of a maximal parabolic subgroup of SL_n which can be described as a block matrix in the upper right-hand corner.

For the rank one cases the character could be conjugated to be supported only on a simple root. We could then expanded along all the columns left of this matrix block, after which we had essentially reached the same situation as for an intermediate step in the proof of theorem III-A such that we could use the same lemmas for expanding along the remaining rows.

For the rank two case on the other hand, a conjugation is not enough to make the characters supported only on the simple roots. We instead used the root exchange lemma by [52] to achieve this, after which we could expand along the remaining matrix elements as before.

Theorem I-9.4 was proven by starting from the integral over $N^{(w)}(\mathbb{A})$ in (4.4) which can be factorised into a part over the unipotent radical N' of the Borel for G' on which the character is generic, and a remaining integral. The former gives a Whittaker coefficient on N' while the latter gives the extra factors seen in (4.22).

Finally, propositions II-1.1, 1.2 and 1.3 were proven by simply comparing the known expressions from [14, 53–55] with the computations in [131].

Chapter 5

Discussion and outlook

We have in this thesis studied automorphic forms on adelic groups $G(\mathbb{A})$ and computed their Fourier coefficients with respect to different unipotent subgroups $U(\mathbb{A})$. These Fourier coefficients carry information about non-perturbative effects in string theory.

Different choices of maximal parabolic subgroups allow for the study of different kinds of non-perturbative effects with respect to different limits of the string moduli space as discussed in section 2.4.4.

In particular, the arithmetic, or p-adic, parts of the Fourier coefficients determine the instanton measure, which, in the ten-dimensional case, sums over the degeneracy of D-instanton states. The results from [43], briefly discussed in section 1.2, matches the known counting of $\frac{1}{4}$-BPS states by a helicity supertrace. This supports the claim that the coefficient functions in the D-dimensional effective action can be used for the counting of BPS black hole microstates in dimension $D+1$ and motivates the importance of the methods developed in this thesis.

Following Paper I we have defined automorphic forms and representations on $G(\mathbb{A})$ and described their Fourier expansions with respect to different unipotent subgroups. In particular, we have studied Whittaker coefficients and how they can be computed using Langlands' constant term formula, the Casselman-Shalika formula and the reduction formula presented in section 4.1.

In accordance with the objectives of this thesis as stated in section 1.3, we have, in Paper II and Paper III, obtained new methods for computing maximal parabolic Fourier coefficients for automorphic forms attached to small automorphic representations of SL_3 and SL_4, as well as SL_n for $n \geq 5$ respectively in terms of highly degenerate Whittaker coefficients. Specifically, we have found that for a minimal automorphic representation of SL_n, a maximal parabolic Fourier coefficient is equal to a maximally degenerate Whittaker coefficient. Additionally, a maximal parabolic Fourier coefficient for a next-to-minimal automorphic representation is determined by Whittaker coefficients supported on at most roots whose Chevalley generators are commuting.

According to propositions II-1.1, 1.2 and 1.3, this seems to also hold for a minimal automorphic representation of E_6, E_7 and E_8, which would greatly simplify the computation not only of the BPS state degeneracies in lower dimensions but also for other non-perturbative contributions in different limits as well.

The methods developed in Paper II and Paper III may also be applied to the D^6R^4 coefficient $\mathcal{E}^{(D)}_{(0,1)}$ even though it is not strictly an automorphic form with an analogous notion of a global wave-front set. From supersymmetry it is expected that such a wave-front set would be associated with nilpotent orbits of type $3A_1$ and A_2 as discussed briefly in section 3.7.1, and more thoroughly in section I-14.1 and [38].

In the future it would be interesting to determine the solutions for the D^6R^4 coefficients for lower dimensions satisfying the differential equation (2.63) and to find a mathematical framework to describe them and the vanishing properties of their Fourier coefficients as they are not strictly automorphic forms.

In table 2.2, we stopped at the dimension $D = 3$, but if we would continue, the remaining groups for lower dimensions are Kac–Moody groups. There has been a lot of recent work on automorphic forms and Eisenstein series on Kac–Moody groups [137–142]. We hope to extend our methods for this setting as well.

There has also been a lot of recent progress for other string theories and compactifications leading to automorphic forms on other groups and several Fourier expansions of interest in string theory have been computed in [143, 144].

5.1 Progress for the exceptional groups

Besides the results already mentioned for the exceptional group in chapter 4, I am also currently working on a project in collaboration with Dmitry Gourevitch, Axel Kleinschmidt, Daniel Persson and Siddhartha Sahi as shown in the List of publications, with the purpose of generalising the methods of Paper III to other simple groups, primarily E_6, E_7 and E_8.

While Paper III relies on matrix manipulations, this project is based on manipulations of Whittaker pairs and their Fourier coefficients directly. This is achieved by deformation of Whittaker pairs, as introduced in [42, 145]. Starting from a neutral Whittaker pair (s, u), we may consider another Whittaker pair (S_t, u) while varying t, where $S_t = s + tZ$ with a rational number $0 < t < 1$ and Z an element commuting with s and u. If $\mathfrak{n}_{S_t} = \mathfrak{n}_{S_{t+\epsilon}}$ for any small rational ϵ, t is called regular, and otherwise it is called critical. That is, at critical t the unipotent subgroup, with respect to which we compute the associated Fourier coefficient, is changed and we may relate the Fourier coefficients at each side of the critical t by further integrations or Fourier expansions in the extra variables. At each critical point we may use [42, Theorem C] as described in section 3.7 to single out only the non-vanishing extra Fourier coefficients.

In this way we plan to iterate over critical points to express a maximal parabolic Fourier coefficient in terms of Fourier coefficients of neutral Whittaker pairs, which we, in turn, would relate to Whittaker coefficients.

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